Relationships between Constructive, Predicative and Classical Systems of Analysis

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Both the constructive and predicative approaches to mathematics arose during the period of what was felt to be a foundational crisis in the early part of this century. Each critiqued an essential logical aspect of classical mathematics, namely concerning the unrestricted use of the law of excluded middle on the one hand, and of apparently circular "impredicative" definitions on the other. But the positive redevelopment of mathematics along constructive, resp. predicative grounds did not emerge as really viable alternatives to classical, set-theoretically based mathematics until the 1960s. Now we have a massive amount of information, to which this lecture will constitute an introduction, about what can be done by what means, and about the theoretical interrelationships between various formal systems for constructive, predicative and classical analysis.

In this final lecture I will be sketching some redevelopments of classical analysis on both constructive and predicative grounds, with an emphasis on modern approaches. In the case of constructivity, I have very little to say about Brouwerian intuitionism, which has been discussed extensively in other lectures at this conference, and concentrate instead on the approach since 1967 of Errett Bishop and his school. In the case of predicativity, I concentrate on developments—also since the 1960s—which take up where Weyl's work left off, as described in my second lecture. In both cases, I first look at these redevelopments from a more informal, mathematical, point

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of view (Part I) and then from a formal, metamathematical point of view (Part II), with each part devoted first to the constructive and then to the predicative redevelopments.

I Informal Mathematical Part

A. Constructive redevelopments of mathematics. In Brouwerian intuitionism the real numbers are treated in some way or another as Cauchy sequences of rationals, understood as (free-) choice sequences. Brouwer's idea concerning these seems to be that one has only a finite amount of information about such sequences at any given time. That was a kind of argument for the continuity conclusion, namely, that any constructive function of choice-sequences must be continuous. Even more:

> *Brouwer's Theorem.* Every function on a closed interval $[a, b]$ is uniformly continuous.

This, on the face of it, is in direct contradiction to classical mathematics, but once it is understood that Brouwer's theorem must be explained differently via the intuitionistic interpretation of the notions involved, an actual contradiction is avoided. Perhaps if different terminology had been used, classical mathematicians would not have found the intuitionistic redevelopment of analysis so off-putting, if not downright puzzling.

In contrast, the Bishop style constructive development of mathematics, which I abbreviate **BCM**—for Bishop Constructive Mathematics, can be read as a part of classical analysis, though developed in more refined terms. This was put on the map by Errett Bishop in 1967 with the publication of his book, *Foundations of Constructive Analysis* (Bishop 1967). Bishop had been working in classical analysis and had made important contributions to that subject over a long period of time. But then he had some radical change of views about classical analysis and felt that it had to be redeveloped on entirely constructive grounds. In a moment I will explain features of his position, as it relates to earlier approaches to constructive analysis. Douglas Bridges joined Bishop in the preparation of a second edition of his book, when Bishop decided that some parts needed reworking, especially the theory of measure. Bridges had published a book on constructive functional analysis in the 1970s (Bridges 1979), and was eminently suited to help in this way. Unfortunately Bishop died of leukemia before the second edition (Bishop and

Bridges 1985) appeared, but Bridges was pretty faithful to Bishop's original conception in completing the work.

Besides these works in constructive analysis, a substantial amount of classical algebra has been redeveloped in the Bishop style approach; the main reference there is *A Course in Constructive Algebra* (Mines, Richman and Ruitenberg 1988).

Bishop criticized both non-constructive classical mathematics and intuitionism. He called non-constructive mathematics a "scandal", particularly because of its "deficiency in numerical meaning". What he simply meant was that if you say something exists you ought to be able to produce it, and if you say there is a function which does something on the natural numbers then you ought to be able to produce a machine which calculates it out at each number. His criticism of intuitionism was its failure, simply, to convince mathematicians that there is a workable alternative to classical mathematics which provides this kind of numerical information (though intuitionistic reasoning also provides that in principle).

General style of BCM. Since, as I said, Bishop's redevelopment of analysis is part of classical analysis, several refinements of classical notions had to be made in order to give it constructive content (or "numerical meaning"). Bishop explained in general terms how this was to be done, using the following dicta.

First of all, use only "affirmative" or "positive" concepts. For example, the inequality relation between real numbers is redefined to mean that you have a rational witness which separates the two numbers by being greater than one and less than the other.

Second, avoid "irrelevant" definitions. For example, the idea of an arbitrary function of real numbers is irrelevant for Bishop because there is nothing useful you can do with it. Instead, he begins by dealing with a very special class of functions of real numbers, namely those which are uniformly continuous on very compact interval. In this way, he finesses the whole issue of how one arrives at Brouwer's theorem by saying that those are the only functions, at least initially, that one is going to talk about. (So, the question is: if you just talk about those kinds of functions, are you going to be able to do a lot of interesting mathematics? That is, in fact, the case!)

Third, avoid "pseudo-generality". An example of avoiding "pseudo-generality" is that Bishop never works on non-separable spaces. Every space he works with is separable—that is, has a countable dense subset. One has the same restriction in predicative mathematics, but Bishop uses it in a special way.

Bishop's language of sets and functions is very close to everyday mathematical language. He does not use the concept of "choice sequence". A sequence, for Bishop, is a sequence, a set is a set, and a function is a function, though in each case with some added structure or constraints. So if a classical mathematician reads Bishop's book he can say: well, I do not see what is very special about this. But what *is* special is the way in which concepts are chosen and the way in which arguments are carried out. Concepts are chosen so that there is a lot of witnessing information introduced in a way that is not customary in classical mathematics, where it is hidden, for instance, by implicit use of the Axiom of Choice. What Bishop does is to take that kind of information and make it a part of his explicit package of what his concepts are up to. We shall see what that means in a moment.

I have a footnote here: Bishop refuses to identify his functions, $f : \mathbf{N} \to \mathbf{N}$, with recursive functions. Nevertheless, as we shall see, there is an interpretation of his language in which the functions on $\mathbf{N}$ are recursive functions. By leaving this open, Bishop's results have generality, so that they can be read by the classical mathematician as applying to arbitrary functions, while the constructive mathematician can read them as applying to computable functions in an informal sense.

Foundations of real analysis in BCM, compared to classical analysis. Classically a Cauchy sequence is simply a sequence $\langle x_n \rangle_n$ of rational numbers $x_n (n \in \mathbf{N})$ such that for any degree of accuracy, $1/p+1$, where p is a natural number, you can get within that by going sufficiently far out in the Cauchy sequence, i.e. such that

$$\forall p\, \exists k\, \forall n,\, m \geq k \left[\; |x_n - x_m| < \frac{1}{p+1} \;\right] \;.$$

Now, for Bishop, a Cauchy sequence is one where you tell how far out in the sequence, $\langle x_n \rangle_n$, you have to go in order to get within degree of accuracy, $1/p+1$. That is given by a function $K(p)$ satisfying

$$\forall p\, \forall n, m \geq K(p) \left[\; |x_n - x_m| < \frac{1}{p+1} \;\right] \;.$$

K is called a *modulus-of-convergence-function* for that sequence. (This is a mild modification of the way Bishop does it.[1]) Now the set $\mathbf{R}$ of real numbers in **BCM** is not defined as the set of Cauchy sequences, but rather as the set of *pairs*

$$(\langle x_n \rangle_n, K) \text{ where } x_n \in \mathbf{Q}$$

and K is an associated modulus-of-convergence-function. It is those pairs that you operate on when you are working with real numbers. You have to have explicit information about how far you need to go out in the sequence in order to be within $1/p+1$ of the answer.

Equality, $=_{\mathbf{R}}$, of real numbers is defined as usual, which means that two real numbers $(\langle x_n \rangle_n, K)$ and $(\langle y_n \rangle_n, L)$ are equal if they have the same limit. But we do not take the real numbers in the classical way to be equivalence classes of Cauchy sequences. What we have to do instead is be sure that when we are dealing with a function on $\mathbf{R}$ as defined above, it preserves $=_{\mathbf{R}}$.

One next has straightforward definitions of addition, subtraction, multiplication and absolute value of real numbers. For example, $(\langle x_n \rangle, K) +_{\mathbf{R}} (\langle y_n \rangle, L) = (\langle x_n + y_n \rangle_n, M)$ where $M(p) = \max(K(2p+1), L(2p+1))$. To compute M for the case of division, $(\langle x_n \rangle, K) \div_{\mathbf{R}} (\langle y_n \rangle, L)$, one must explicitly incorporate a bit of information q that shows the limit of the $|y_n|s$ to be at least $1/q+1$ (where $q \in \mathbf{N}$).

Bishop makes systematic use of sets A with an equivalence relation $=_A$ on them, rather than the corresponding sets of equivalence classes, and also of functions preserving the equivalence relations rather than functions on the equivalence classes. In that way you can truly talk about numerical or computational implementation of his notions. For example, a computational implementation of a function on real numbers will take for each argument a sequence of rationals which is given computationally, and a function K that is given computationally, with both given by algorithms as data for which we compute the value of the function, represented by certain output data. We might have a different presentation of that same real number and we will get a computation which gives a different answer in the way it is represented but which has to be equal to it in the sense of equality on the real numbers.

Taking the definition of real numbers to be modified in this way, you end up with a constructive version of the real number system. For instance, a form of the *Cauchy Completeness Theorem* holds: every Cauchy sequence

[1]He takes these to be sequences $\langle x_n \rangle_n$ such that $\forall n, m \geq 1 \ \left[\, |x_n - x_m| < \frac{1}{n} + \frac{1}{m} \right]$, so that $K(p) = 2(p+1)$ works in this case.

of reals with a modulus-of-convergence function will converge to a real in Bishop's sense.

What is a continuous function? Classically, such $f : [a,b] \to \mathbf{R}$ on a closed interval is a function satisfying

$$\forall x \in [a,b]\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in [a,b]\ \{\ |x-y| < \delta \to |f(x) - f(y)| < \varepsilon\ \}$$

where ε and δ are rational numbers. And, again classically, a uniformly continuous function, $f : [a,b] \to \mathbf{R}$, is one where we can give δ uniformly in terms of ε, independently of where x and y are in the interval $[a,b]$, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in [a,b]\ \{\ |x-y| < \delta \to |f(x) - f(y)| < \varepsilon\ \}\ .$$

Classically we have that if a function is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$. This goes back to the Heine-Borel Covering Theorem.

Now, as explained above, Bishop starts out by saying that we are not going to talk about arbitrary functions, we are only going to talk about uniformly continuous functions on compact intervals. He thus defines the class of functions from the closed interval $[a,b]$ into real numbers, $C([a,b], \mathbf{R})$, to consist of all pairs (f, D), where $f : [a,b] \to \mathbf{R}$ is a function which preserves the equality $=_{\mathbf{R}}$ on the real numbers and for which $D : \mathbf{Q}^+ \to \mathbf{Q}^+$ is a function which tells you how δ depends uniformly on ε. That is, D is a *modulus-of-uniform-convergence function* for f, in the sense that

$$\forall \varepsilon > 0\ \forall x, y \in [a,b]\ \{\ |x-y| < D(\varepsilon) \to |f(x) - f(y)| < \varepsilon\ \}$$

This is generalized to more abstract classes of spaces. To begin with, Bishop works with separable metric spaces, and he defines compactness for these in a very particular way as follows. A metric space is called *totally bounded* if for every ε it can be covered by a finite number of ε-neighborhoods. That means that given any ε you can find a finite set of points $x_1, \ldots, x_n$ such that any point of the space is within a distance less than ε of one of these points, i.e. you have a function of ε which actually produces the required points. Then a separable metric space is defined to be compact if it is totally bounded and every Cauchy sequence (in his sense) converges. Finally, the function spaces $C(X, Y)$ are defined whenever X is compact in this sense and Y is a separable metric space. Again that is the class of uniformly continuous functions $f : X \to Y$ with a uniform-modulus-of-continuity function given as witnessing information.

On the basis of this kind of systematic refinement of classical concepts, the **BCM** school redevelops substantial tracts of 19th and 20th century analysis and algebra constructively. To be fully convinced of that you just have to go through the expositions referred to above to see for yourself. It is an impressive body of work, and it is not off-putting to the working (classical) mathematician in the way that Brouwerian intuitionism was. If you are willing to be interested in the development at all as a mathematician, you can read it, and you can take in what all these concepts are. It does not conflict with your ordinary mathematical feelings about what these notions are, and once you get the style of using witnessing information systematically, you see how it goes and you get a feeling for why it is constructive. Moreover, you can see the relation with classical mathematics in the following way. Bishop formulates what he calls the *Limited Principle of Omniscience*, **LPO**, about natural numbers, as follows

LPO $\quad \forall n(f(n)=0) \vee \exists n(f(n) \neq 0)$

which is just a special case of the *Law of the Excluded Middle.* Bishop asserts that each theorem of **BCM** is a constructive substitute ϕ^* for a classical theorem ϕ, and you can get back the classical theorem, ϕ, from the constructive version ϕ^*, simply by adding the principle **LPO**, i.e.

$$\mathbf{LPO} \wedge \phi^* \rightarrow \phi$$

In that sense too, **BCM** is just a refinement of classical mathematics, a refinement which in an intuitive sense has constructive content. What that comes to from a logical point of view will be taken up in Part IIA below. Before that, we turn next to comparisons with the predicative program.

B. Predicative redevelopments of classical mathematics. Although a great deal of work has been done since the 1960s as a continuation of Weyl's program there are, unfortunately, no texts one can point to for a systematic exposition, at least none comparable to those referred to above for **BCM**. One book that people mention in this respect is Paul Lorenzen's *Differential und Integral* (Lorenzen 1965); while significant portions of that are based on predicative grounds, it is not restricted to such. The monograph of Gaisi Takeuti, *Two Applications of Logic to Mathematics* (Takeuti 1978) is, on the other hand, clearly predicative; that presents a finite-type extension of the system $\mathbf{ACA}_0$ (defined in my second lecture), and shows how various parts of classical analysis can be formalized there. For predicative developments of classical *and* modern analysis, one can point to substantial portions of Stephen Simpson's book, *Subsystems of Second Order Arithmetic* (Simpson 1998); see especially Chs. III and IV. I shall be concentrating in the following, instead, on my own approach which has been outlined in several articles (references below) and elaborated in unpublished notes.

Before going into that, a brief comparison with the work in the Reverse Mathematics program established by Harvey Friedman and carried on by Steve Simpson and his students, is in order; that is what (Simpson 1998) is devoted to. In the Reverse Mathematics program one studies certain second order formal systems, such as $\mathbf{ACA}_0$, where you have variables for natural numbers and variables for sets of natural numbers, or for functions of natural numbers. (Concepts of analysis such as real numbers are naturally represented at the second-order level, but things become a bit more awkward as soon as one ascends to various kinds of sets and functions of real numbers and function spaces used in analysis.) It is shown in the Reverse Mathematics program that there are five basic set existence principles from which many results of classical and modern analysis, topology, and algebra follow. It happens that many of these results are equivalent to the basic set existence principles from which they follow, i.e. the implications can be *reversed*, and that is the main concern in Reverse Mathematics.

By comparison, my own main interest is in *consequences* rather than *equivalences*, and my concern is to have formal systems justified on basic grounds of one kind or another, in which the mathematics in question can be developed in as direct a way as possible; this generally means going beyond second-order systems. In particular, I have developed such systems in which you can redevelop substantial portions of classical and modern analysis on predicatively justified grounds. One such is a system I call **W**, in honor of Weyl, which is in a certain sense a variable finite type extension of $\mathbf{ACA}_0$. I will describe that later in the final logical part of this article.

In **W** you can decide, in a sense that I will describe, questions of the form: $\exists n f(n) = 0$; in other words the Limit Principle of Omniscience, **LPO**, holds. Moreover, all the work of (Bishop and Bridges 1985) can be directly represented within **W**. Recall that for each classical theorem ϕ considered in the Bishop and Bridges text, a constructive substitute ϕ^* is found such that $\phi^* \wedge \mathbf{LPO}$ implies ϕ. So you could say: let us put these two together. We go to their book, see all the theorems ϕ^* that they get, add **LPO**, which is in **W**, and thus obtain the classical theorem ϕ in **W**. There is of course an immediate conviction about how much can be done in this way, but we might like to see—without going through this detour which involves additional complexity of various kinds—just what can be done directly in **W**. So we start again: how do we treat the real numbers, how do we treat functions of real numbers, and so on through the whole business. As I said in Lecture 2, the way Weyl presented the real numbers was via Dedekind sections. But it is actually more convenient to work with Cauchy sequences. Unlike the approach in **BCM** where a modulus-of-convergence function is part of what constitutes a Cauchy sequence, in the system **W**, these are defined just as they are classically. That follows from the fact that the formula

$$\forall n, m \geq k \left[\, |x_n - x_m| < \frac{1}{p+1} \, \right]$$

is arithmetical, hence it can be decided by **LPO**, and the least such k can be determined without building it in as additional information. Thus, I do not have to change the definition of Cauchy sequence from its classical definition.

Among the theorems that you can prove in **W** are the *Bolzano-Weierstrass Theorem* saying that the real numbers are locally sequentially compact. In particular it follows that **R** is Cauchy complete, because any Cauchy sequence is bounded, therefore it has a convergent subsequence and therefore it must converge. However, we cannot prove the LUB axiom for sets in **W**, for the same reason, basically, that it is not provable in $\mathbf{ACA}_0$.

The argument for the Bolzano-Weierstrass Theorem is by subdivision: if we have an infinite sequence in the interval $[a, b]$ then divide this interval in half and ask if there are infinitely many terms of the sequence which lie in the left-hand half; if there are, we are then going to continue on that left-hand half; if not, then there must be infinitely many terms on the right-hand half and we will work on the right-hand side. Whichever half we choose, we divide it again, and ask which half has infinitely many, and so on. But the question whether "there are infinitely many ϕ's" is of the form

$$\forall n \, [\, \phi(n) \rightarrow (\exists k > n) \, \phi(k) \,] \, .$$

That is an arithmetical formula when ϕ is arithmetical and it is then a question which is decided by **LPO**. Hence, the subdivision argument proceeds in **W** as it does classically.

Beyond **R**, as in **BCM**, we work within separable metric spaces, where again a countable sub-basis is explicitly given. One then verifies that many familiar spaces such as finite dimensional real and complex spaces, as well as

Baire space, are locally sequentially compact. One of the consequences for such spaces is the *Stone-Weierstrass Theorem*, which tells us that arbitrary continuous functions can be approximated in a very nice way.

It was not obvious how to do measure theory in Weyl's setup. Standard classical presentations of measure theory start out with the definition of measure of a measurable subset, say, of a finite interval, via the definition of outer measure. Now, outer measure looks at the shrinking of open covers of the given set and takes the greatest lower bound of the measures of those open sets. We can define measures of open sets of reals very nicely because they decompose into a countable union of disjoint open intervals. But for outer measure we must then apply the GLB axiom, which is not provable in **W**. Instead, measurable sets are there taken to be ones that are well approximated, both themselves and their complements, by sequences of open covers. You simply use the approximations from *both* sides in order to deal with measurable sets directly without going through outer measure. It turns out in this development that we cannot prove the existence of non-measurable sets. It is consistent with **W** that all sets of real numbers are measurable, but we do not assume that. What can be done in **W** is a kind of "positive" development of measure theory.

Following that, one develops the theory of Lebesgue integration, measurable functions, and Lebesgue integrable functions and proceeds into functional analysis in a fairly standard way. One can deal in **W** with linear operators on separable Banach and Hilbert spaces and carry out the spectral theory of bounded operators. All of this has been worked out by me in unpublished notes. One further thing I started to work on was how to do the spectral theory of unbounded operators. One way of doing that is to approximate unbounded operators by bounded operators in a systematic way, and that looked like it should go through.[2]

As a result of such work, I proposed in (Feferman 1988, 1993) the following

Conjecture All (or almost all) scientifically applicable analysis can be carried out in **W**.

Of course, that is by no means all of analysis. There is a lot of analysis which goes beyond any potential scientific application, including analysis on non-

[2]While preparing the texts of these lectures for publication, I learned of relevant Ph.D. thesis work by Feng Ye at Princeton University. According to this work, he shows, among other things, how to develop the spectral theory of unbounded operators in a system based on finitary constructive reasoning that is much weaker than **W**.

separable spaces and analysis which involves non-measurable sets, neither of which can be done in **W**. So one must consider test cases in the applications of analysis to science in which those kinds of things might figure. For example, Itamar Pitowsky has proposed a use of non-measurable sets in quantum mechanics, but there is considerable dispute as to whether that is a reasonable model. Possible uses of non-separable spaces in other aspects of quantum theory have been suggested by G. G. Emch—and again those are very speculative models, which are by no means generally accepted.[3] Otherwise my working conjecture is at least corroborated in the settled scientific applications of analysis.

II Metamathematical Part

Now I want to turn to the metamathematics of both constructive and predicative systems of the sort I've just described. Let us start by going back to Bishop Constructive Mathematics (**BCM**) and see what kinds of things are available here.

A. Formal systems for BCM. I shall concentrate on two papers of my own, *A language and axioms for explicit mathematics* (Feferman 1975) which covers both constructive and predicative mathematics and, more specifically for constructive mathematics, *Constructive theories of functions and classes* (Feferman 1988). But let me refer you also to a few other approaches to formal systems for **BCM**, including: H. Friedman, *Set-theoretic foundations for constructive analysis* (Friedman 1977), and P. Martin-Löf, *Intuitionistic Type Theory* (Martin-Löf 1984). As to the latter, Martin-Löf came to his concepts of intuitionistic type theory on more or less philosophical grounds; he was not motivated by the question whether you could formalize **BCM** in it, and I do not think he was particularly concerned about that. But people have said that it is in fact one way in which you can look at **BCM** and represent it formally. A lot of information can be found in the book by Michael Beeson, *Foundations of Constructive Mathematics* (Beeson 1985) comparing different formal approaches, including the ones that I have mentioned. A very good further source, the two volumes by A.S. Troelstra and D. van

[3] In a postscript to my 1998 paper in (Feferman 1998), pp. 281–283, I have presented some negative evidence concerning the Emch and Pitowsky proposals in the literature and from further discussions with several mathematical physicists and applied mathematicians.

Dalen, *Constructivism in Mathematics I, II,* (Troelstra and van Dalen 1988) also presents various different approaches. Naturally, I favor my own, and I will sketch that now.

What I did in (Feferman 1975) was to introduce some formal systems of "Explicit Mathematics", to begin with, a theory $\mathbf{T}_0$ which is constructive in a suitable sense of the word, and then an extension $\mathbf{T}_1$ which incorporates predicative systems. The paper (Feferman 1979) elaborates on the uses of $\mathbf{T}_0$ and its metamathematical properties. We conceive its universe of discourse, V to be rather rich: it includes the natural numbers, is closed under pairing, and includes elements which are regarded as partial functions. Then partial functions can apply to natural numbers or n-tuples of natural numbers or n-tuples of other objects of the universe, and they can also apply to themselves. But then you have the possibility of functionals, because functions applied to functions are simply functionals. There are some basic axioms which govern how these work, the *Applicative Axioms*, **APP**, which are essentially like the axioms for Lambda Calculus, but modified to a form where we are dealing, not with total functions, but with partial functions. So it is a *Partial Applicative Lambda Calculus.* There is a range of models M of **APP** from the recursive to the set-theoretical. In the former case, V is taken to be the natural numbers and functions are taken to be (codes for) partial recursive functions. In the latter case, V is taken to be the cumulative hierarchy and functions are generated from ordinary set-theoretic functions to satisfy **APP**. Every model M of **APP** can be expanded to a model of the remaining axioms of $\mathbf{T}_0$, all of which concern classes.

We deal with classes either regarded as elements of V in an intensional way or as represented by elements of V. In the latter case they are named by elements of V, so we can operate constructively on the names, and we may regard the classes themselves as extensional objects. There is a choice there, but it is an inessential difference, it is just a formal difference whether you take them at the outset as intensional objects, that is as predicates, or as extensional objects represented in possibly different ways. For these you have an *Elementary Comprehension Axiom* (**ECA**) which is elementary in the sense that you do not have quantifiers ranging over classes in the statement of which classes are asserted to exist, only over the elements of the universe, V. But since V is very rich you may have quantification not only over numbers but also over partial functions in **ECA**.

Examples of constructions of classes which follow from **ECA** are: $X \times Y$, X^n and $X \to Y$ (the class of all functions which are total on X to Y.) In addition to **APP** and **ECA**, the system $\mathbf{T}_0$ further contains: usual axioms for the class of natural numbers **N**, including the full induction scheme; inductively generated classes in general (**IG**), and a *Join Axiom* (**J**) which allows us to perform the following operations on sequences (or families) $\langle B_x \rangle_{x \in A}$ of classes,

$$\bigcup_{x \in A} B_x, \quad \bigcap_{x \in A} B_x, \quad \sum_{x \in A} B_x, \quad \prod_{x \in A} B_x,$$

among others. It turns out that it is sufficient in **J** to posit the disjoint union construction $\sum_{x \in A} B_x$, from which the others are constructed by **ECA**.

All of **BCM** may be comfortably formalized in $\mathbf{T}_0$. But that system is proof-theoretically very strong, and goes far beyond what is needed to do so. For that purpose, I introduced a relatively (proof-theoretically) weak subsystem $\mathbf{EM}_0 \restriction$ of $\mathbf{T}_0$ in the 1979 paper referenced above. It omits the **J** and **IG** axioms, and restricts the induction scheme for **N** to classes, as in $\mathbf{ACA}_0$; that is called *Restricted Induction*, as is indicated by the sign $\restriction$. ('**EM**' is just an acronym for 'Explicit Mathematics'.) Though the join operation is not available in this system, the above operations on sequences of classes $\langle B_x \rangle_{x \in A}$ can still be carried out for "pre-joined" families, i.e. for which the class $\{(x, y) \,|\, y \in B_x\}$ is given in advance.

These systems are presented within the classical two sorted predicate calculus, but, if we want to, we can certainly consider intuitionistic versions by omitting the Law of the Excluded Middle; then we put an ^{i} to indicate that: $\mathbf{T}_0^i$, $(\mathbf{EM}_0 \restriction)^i$.

The main meta-theorems obtained for $\mathbf{EM}_0 \restriction$, are first of all that in classical logic the system is a conservative extension of Peano Arithmetic, **PA**; I showed that by a simple model-theoretic argument. Then for intuitionistic logic, Beeson showed by using Kripke-models, instead, that $(\mathbf{EM}_0 \restriction)^i$ is a conservative extension of Heyting Arithmetic, **HA**, which is **PA** without the Law of the Excluded Middle.

Let us go back to what Bishop and Bridges did in their revised version of Bishop's book and ask: how much of it can be formalized in this system? They never use the Law of the Excluded Middle, so we can ask how much of it can be formalized in $(\mathbf{EM}_0 \restriction)^i$. It simply comes down to the questions: where would full induction on **N** and the use of non-pre-joined families and general inductively generated classes come in to the picture? These last are used in $\mathbf{T}_0$ for a constructive theory of ordinals as constructive tree-classes. Now in the original version of Bishop's book, he did measure theory by using

Borel classes, and Borel classes in effect require using ordinals. But, later, he was able to avoid the use of Borel classes by a quite different approach he worked out with a student of his (H. Cheng), and that is what you find in Bishop and Bridges' book. It replaces the theory of Borel classes and therefore does not use ordinals, and thus does not require **IG**.

By contrast, in Bishop style constructive algebra there is a part of Abelian group theory where ordinals do come into the picture in an essential way, namely in what is called the Ulm theory of countable Abelian groups, and that definitely cannot be represented in $(\mathbf{EM}_0\!\restriction)^i$ (though it can be in $\mathbf{T}_0$). But ordinary finitely generated Abelian group theory all goes through without the use of ordinals. Aside from that, coming back to possible uses of full induction on **N** and full Join axiom in (Bishop and Bridges 1985), one just has to examine these case by case. By being sensitive to that question and looking at various test cases you can see that all of **BCM** that does not use ordinals can in fact be formalized in $(\mathbf{EM}_0\!\restriction)^i$ and therefore rests on a basis that does not assume any more than Heyting arithmetic.

B. Formal systems for predicativity. To conclude, let us look at the metamathematical picture for predicativity. Here there are quite a few references going back to the mid-60s, beginning with the work of myself (Feferman 1964) and K. Schütte (1965) on the analysis of the full extent of predicative mathematics. For further references, see (Feferman 1987) and (Feferman and Jäger 1993, 1996).

To go back to the work of Schütte and myself from 1964, what this dealt with was Kreisel's proposal to characterize what is predicative via transfinite ramified analysis. Although that is not a suitable foundation for the actual predicative development of analysis, conceptually it is the appropriate place to look at predicativity, though not through arbitrary ordinals, which would not make sense predicatively. Only those ordinals which one can access step by step from below by a kind of *autonomy* or *bootstrap condition* are to be considered. What Schütte and I achieved independently in 1964 was a determination of the least ordinal that is not obtainable in this way. And that ordinal is the limit Γ_0 of the Veblen hierarchy of critical functions described in Lecture 1. The union of systems of ramified analysis up to but not including Γ_0 is denoted $\mathbf{RA}_{<\Gamma_0}$. Now if you want to see what part of analysis can actually be carried out on predicative grounds, one needs subsystems which are predicatively justified, but which are not ramified. And by predicative justification of a system is meant a proof theoretic reduction to $\mathbf{RA}_{<\Gamma_0}$. For example, $\mathbf{ACA}_0$ is predicatively justified in this sense, since it can be interpreted in $\mathbf{RA}_0$, ramified analysis at the lowest level. The system $\mathbf{W}$ discussed informally in Part IB is a candidate for a more flexible predicatively justified system which is also unramified. To spell out its principles, first of all $\mathbf{W}$ is contained in the system $\mathbf{T}_1$ of (Feferman 1975). One axiom of $\mathbf{W}$ says explicitly that existential quantification over natural numbers is something that is decided by a functional μ, as follows:

$$(\mu) \qquad \exists n f(n) = 0 \rightarrow f(\mu(f)) = 0 \ .$$

Here we can think of $\mu(f)$ as the least n such that $f(n) = 0$, if there is such an n, otherwise it takes the value 0, and so μ is called the *non-constructive minimum operator*. As in the system $\mathbf{EM}_0\!\restriction$, we make use of a restricted form of induction on $\mathbf{N}$. There it was restricted to classes, while in $\mathbf{W}$ we have a still more restricted form called *function induction*. It says that if a function f at 0 is 0 and if the property $f(x) = 0$ is closed under successor, then f is 0 at all natural numbers:

$$(\mathit{Fun\text{-}Ind}_{\mathbf{N}}) \qquad f(0) = 0 \wedge \forall x(f(x) = 0 \rightarrow f(x') = 0) \rightarrow \forall x \in \mathbf{N}(f(x) = 0) \ .$$

Equivalently: if two functions agree at 0 and whenever they agree at x they agree at its successor then they agree on all natural numbers. With these modifications, the system $\mathbf{W}$ is very similar to $\mathbf{EM}_0\!\restriction$. It has the *Applicative Axioms*, but now beefed up with the functional, μ. It has the basic axioms for 0 and of successor on the natural numbers ($\mathbf{N}$-*Axioms*) as before, but in place of Class Induction it has Function Induction; finally it has Elementary Comprehension Axiom, as before. Symbolically,

$$\mathbf{W} = \mathbf{APP} + (\mu) + \mathbf{N}\text{-}\mathit{Axs} \ + (\mathit{Fun\text{-}Ind}_{\mathbf{N}}) + \mathbf{ECA} \ .$$

The main metamathematical result for $\mathbf{W}$ is due to Jäger and myself (Feferman and Jäger 1993, 1996); we show (by a much more difficult argument than for $\mathbf{EM}_0\!\restriction$) that $\mathbf{W}$ is a conservative extension of $\mathbf{PA}$. Hence, it also is predicatively reducible.

From our metatheorem and the working hypothesis that all (or almost all) scientifically applicable analysis can be carried out in $\mathbf{W}$ it follows that the part of mathematics needed for science rests on completely arithmetical grounds. The significance of this is discussed further, in the conclusion below.

It is natural to ask whether still weaker systems (proof-theoretically) serve the same purpose. That has been established to an extent in Reverse Mathematics. There, as mentioned before, only second order systems are used; among these, one second order system emerges that comes up when we are dealing with the subdivision argument. It is based on the so-called *Weak König's Lemma*, **WKL**. Full **KL** concerns arbitrary finitely branching trees, where **WKL** simply deals with binary branching trees. Both say, classically, that if there are no infinite branches then there is a common finite bound on the length of all the branches, or equivalently, if there are arbitrarily long branches then there is an infinite branch. The system $\mathbf{WKL_0}$ treated in Reverse Mathematics is obtained from $\mathbf{ACA_0}$ by restricting the set existence axiom **ACA** to recursive predicates and adding the statement **WKL**. It was shown by Friedman by a model-theoretic argument and then by Sieg (1991) by a proof-theoretical argument using Herbrand-Gentzen style methods, that $\mathbf{WKL_0}$ is a conservative extension of Primitive Recursive Arithmetic, **PRA**. The system **PRA** is a fragment of **PA** that is purely quantifier-free and simply has the usual defining axioms for primitive recursive functions and a quantifier-free rule of induction. It has been argued by Tait that **PRA** represents exactly finitistic mathematics. Though what finitism consists in is not a settled matter, Tait's thesis is generally granted.

So now the questions is, how much of classical and modern analysis can already be carried out in $\mathbf{WKL_0}$? Simpson and his students have gone through this and shown that, not only the analysis of step-wise continuous functions can be done there, but also substantial portions of functional analysis can be handled there as well. (Cf. (Simpson 1998), Ch. IV for a detailed exposition and further references).

Since, in my view, flexible (variable) finite type systems are preferable to second order systems when examining such questions, let us go back to the system **W**. The obvious conjecture is that if we replace the (μ)-axiom by its consequence **WKL** in the system **W**, we then obtain a conservative extension of **PRA**. That would give a system in which substantial portions of scientifically applicable analysis can be formalized and which can, in principle, be justified on finitistic grounds. (The argument that Jäger and I gave for **W** does not pull down directly to this subsystem of **W**. So, the matter has to be looked at again.)

To conclude, in (Feferman 1993) I have discussed the significance of my working hypothesis concerning scientifically applicable mathematics in **W**—which as we have seen is a conservative extension of **PA**—for the so-called

Quine-Putnam indispensability arguments. Their thesis was, as summarized by Penelope Maddy (cf. op. cit. for the source): "We have good reason to believe our best scientific theories, and mathematical entities are indispensable to those theories, so we have good reason to believe in mathematical entities. Mathematics is thus on a . . . par with natural science [and] the evidence that confirms scientific theories also confirms the required mathematics." Quine argued that this justifies Zermelo set theory, **Z**, but not Zermelo-Fraenkel set theory, **ZF**. The reasoning was that we need the Axiom of Infinity to obtain the natural numbers **N** and then the Power Set Axiom to obtain the real numbers **R**, and its application once more to obtain the set of real functions $\mathbf{R} \rightarrow \mathbf{R}$; according to Quine, one is led to accept something like Zermelo set theory **Z** by a "simplificatory rounding out," though no more is necessary for actual science. But **Z** is both highly impredicative and vastly stronger than even full second order analysis, which in turn is impredicative and vastly stronger than the kinds of systems such as **W** we have been dealing with here. Quine's acceptance of **Z** is based on an uncritical examination of what is actually needed in mathematics for natural science. The work described here shows that, as least as far as currently applicable mathematics is concerned, we do not need to go beyond systems of strength **PA**. I, myself, do not accept the indispensability arguments but think it is philosophically important to be aware of that result if one accepts them at all.

This concludes my tour of those modern approaches to constructive and predicative mathematics, and of associated formal systems, with which I am most closely acquainted. Whatever approach one prefers (and the references below can be pursued for other such), I hope you are convinced by the work presented here of both the viability of constructive and predicative alternatives to classical mathematics (at least of its scientifically applicable part) as well as of my slogan that "a little bit goes a long way".

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