1987 :

Dear Millson,

Nori brought me a copy of your paper with Goldman on deformations of flat bundles. It seems to me that it contains a number of inaccuracies but also that, when corrected, it gives the stronger results you ask for p 8 l -10 à -7. I begin with the latter.

The philosophy, which I had not realized before reading your paper, seems the following: in characteristic 0, a deformation problem is controlled by a differential graded Lie algebra ($d^\circ \geq 0$), with quasi-isomorphic DG Lie algebras giving the same deformation theory. If the DG Lie algebra controlling a problem is "formal", ie quasi isomorphic to $\oplus H^i$, then the versal (formal) deformation space is that of $\oplus H^i$, ie is the (completion at 0 of the) subscheme of H^1 defined by the equations $[u,u]=0$.

If one believes in this, what you do becomes transparent:

<u>Deformation problem</u>: G is a given reductive group, and one wants to deform a given G-torsor P. Meant: $G(\mathbb{C})$ torsor, or, equivalently, $G(\mathcal{O})$-torsor, with integrable connection ∇.

<u>Controlling DG algebra</u>: (one is over $\mathbb{C}$):

$$\Omega^{**}(\mathrm{Lie}\, G^P)$$

If P can be reduced to a maximal compact $K \subset G$ (ie to a compact form of G), Hodge theory shows that the controlling algebra is "formal" : one has quasi-isomorphisms of DG Lie algebras

$$(1) \qquad \Omega^{**}(\mathrm{Lie}\, G^{P}) \hookleftarrow \mathrm{Ker}(d') \longrightarrow \frac{\mathrm{Ker}(d')}{\mathrm{Im}(d')}$$

and $\mathrm{Ker}(d')/\mathrm{Im}(d')$ has $d=0$. To check the above, it is easiest to use the principle of the 2 types in the form that the double complex $\Omega^{**}(\mathrm{Lie}\, G^{P})$ is (forgetting $[\,,]$) a sum of

a) a double complex with $d' = d'' = 0$ ("harmonic part")

b) squares of isomorphisms:
$$\begin{tikzcd} * \arrow[r, "\sim"', "d'"] & * \\ * \arrow[u, "d''\wr"] \arrow[r, "\sim", "d'"'] & * \arrow[u, "\wr d''"'] \end{tikzcd} \qquad (0 \text{ elsewhere})$$

As far as formal deformations are concerned, this concludes the story. It is not clear to me how to use Artin in the analytic case [sol. of anal. eqn.] to get the same for analytic-versal. The algebraic case is OK, but Artin is more powerful there.

Here is how to give a meaning to the philosophy (and check it)

<u>Construction</u>: How to attach to a DG Lie algebra L^* (over a char. 0 field k) a fibered category over the category of $Sp(A)$, A local k-algebras of finite dimension with residue field k.

<u>Object over $Sp(A)$</u> : $[k = A/m]$

$$\omega \in L^1 \otimes m, \quad \text{such that} \quad d\omega + \tfrac{1}{2}[\omega,\omega] = 0$$

<u>Map</u> : The Lie algebra L^0 give rise to a formal group $\hat{\underline{L}}^0$; the A-points of this formal group are $L^0 \otimes m$, viewed as a nilpotent Lie algebra, and equipped with Campbell-Hausdorff group law.

Notation : $\hat{\underline{L}}^0(A)$.

The formal group $\hat{\underline{\underline{L}}}^0$ acts on L^1 (thanks to $[\ ]: L^0 \otimes L^1 \to L^1$), hence $\hat{\underline{\underline{L}}}^0(A)$ on $L^1 \otimes A$. One has also a map

$$\text{"}g^{-1}dg\text{"} \ : \ \hat{\underline{\underline{L}}}^0 \longrightarrow L^1 \qquad : \ \exp(\lambda) \longmapsto \frac{1-\exp(-\operatorname{ad}\lambda)}{\operatorname{ad}\lambda}\left[d\lambda\right]$$

$$\text{ie} \quad : \quad \hat{\underline{\underline{L}}}^0(A) \longrightarrow L^1 \otimes m$$

One now define a map from $\omega \in L^1 \otimes m$ to $\omega' \in L^1 \otimes m$ as being $g \in \hat{\underline{\underline{L}}}^0(A)$ such that

$$(2) \qquad \omega' = g\, dg^{-1} + g(\omega)$$

One has to check that $d\omega' + \frac{1}{2}[\omega',\omega'] = g\left(d\omega + \frac{1}{2}[\omega,\omega]\right)$ when ω and ω' are related by (1).

<u>To be checked</u> : for a square zero ideal I in A, and an object ω on A/I, there is an ~~obs~~ obstruction in $H^2(L^{\cdot}\otimes I)$ to extend ω to A; if the obstruction vanish, isomorphism classes of extensions form a principal homogeneous space under $H^1(L^{\cdot}\otimes I)$; automorphisms of any extension (trivial on A/I) are $H^0(L^{\cdot}\otimes I)$. This is functorial in $L^{\cdot}$, hence the fact that quasi-isomorphisms induce equivalences of fibered categories.

<u>Application here</u> Let X be an analytic variety and (V,∇) be a (holomorphic) vector bundle with (holomorphic) $R=0$ connection. The deformation problem is

$A \longmapsto$ category of vector bundles with connection in the X direction, on $X \otimes Sp(A)$, deforming (V,∇).

It is given by the previous construction aplied to $\Omega^{**}(\operatorname{End} V)$.

<u>Rmq</u>: In your application, because you have the explicit quasi-isomorphism (1), you have an explicit equivalence of fibered categories between

* deformations of (V, ∇) as above
* category with objects over $\mathrm{Sp}(A)$ the pointed maps

$$\mathrm{Sp}(A) \longrightarrow \text{subscheme of } \hat{H}^1L \text{ with } [u,u]=0$$

and arrows: $g \in \underline{\underline{(H^0L)}}^{\wedge}(A)$ from u to $g(u)$

The functor $\uparrow$ is such: a cohomology class $u \in H^1L \otimes m$, ~~ditto~~ ie $u : \mathrm{Sp}A \to$ scheme H^1L, can be uniquely written as $\omega \in L^1 \otimes m$ with $d'\omega = d''\omega = 0$. This ω can be corrected by a $d'\ell^0$ to have zero curvature, if $[u,u]=0$. The corrected $\tilde{\omega}$ is unique mod $\exp(\mathrm{Ker}\, d' \otimes m)$.

<u>Rmq</u> (1) is a filtered quasi-isomorphism, for the Hodge filtration. This should translate into saying that the pair of spaces

$$(\text{déformations of } (V,\nabla)) \supset (\text{deformations of } \nabla \text{ on a fixed } V)$$

is isomorphic to ~~the intersection of [illegible] with~~ or the pair

$$(H^1 \supset F^1H^1) \cap \text{locus } [u,u]=0.$$

II. Comments on your paper :

p1 l2 after cor B : no [take $G = U_1$]

p6 l18 holomorphic ? : I guess you want to start with η such that

$$d'\eta = d''\eta = 0$$

p7 l4 : why closed $\Rightarrow$ (2,0)-component closed ?

III Generalisation

Hodge theory is available for variations of Hodge structures (complex variations, on compact Kähler manifolds). The useful bigrading of $\Omega^{**}(\mathrm{End}\,V)$ is obtained by mixing that of Ω and that of $\mathrm{End}\,V$. It is useful now that (1) is a filtered quasi-isomorphism. One gets

a) the same result as before on deformations of a representation of π_1 given by a complex variation

b) looking at the deformation theory controlled by $F^0(\dots)$: same result for the versal moduli of a filtered bundle with connection coming from a complex variation. One insist that the deformed connection must continue to satisfy "transversality" : $\nabla F^i \subset \Omega^1 \otimes F^{i-1}$.

Yours sincerely

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