EQUIVALENCE OF NORMS

KEITH CONRAD

1. Introduction

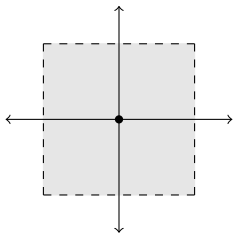
Let K be a field and $|\cdot|$ be an absolute value on K, such as the usual absolute value on $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$, the p-adic absolute value on $\mathbf{Q}$ and on the p-adic numbers $\mathbf{Q}_p$, or the trivial absolute value on a general field K. For a K-vector space V, with $|\cdot|$ understood, a vector space norm on V is a function $\|\cdot\| : V \times V \to \mathbf{R}$ such that

(1) $\|v\| \geq 0$ for all $v \in V$, with equality if and only if $v = 0$.
(2) $\|v + w\| \leq \|v\| + \|w\|$ for all v and w in V (triangle inequality),
(3) $\|cv\| = |c|\,\|v\|$ for all $c \in K$ and $v \in V$. Note the absolute value on K plays a role here.

A norm $\|\cdot\|$ on V defines a metric on V by $d(v, w) = \|v - w\|$, from which we get open subsets and closed subsets of V relative to $\|\cdot\|$. Different norms always define different metrics since the norm can be recovered from the metric it defines ($\|v\| = d(v, 0)$), but different norms might define the same concept of open subset of V.

Example 1.1. On $\mathbf{R}^2$ let $\|\cdot\|_2$ be the Euclidean norm: $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$. Another norm on $\mathbf{R}^2$ is $\|v\|' = 2\,\|v\|_2$. The condition $\|v - a\|' < r$ is the same as $\|v - a\|_2 < r/2$, so open balls in $\mathbf{R}^2$ defined by $\|\cdot\|_2$ and $\|\cdot\|'$ are the same (even if the radii don't match). Open subsets for a norm are unions of open balls for that norm, so the open subsets of $\mathbf{R}^2$ for $\|\cdot\|_2$ and $\|\cdot\|'$ coincide.

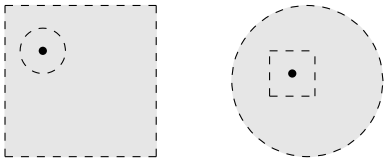
Example 1.2. On $\mathbf{R}^2$ let $\|\cdot\|_2$ be the Euclidean norm and set $\|(x, y)\|' = \max(|x|, |y|)$. It is left to the reader to check $\|\cdot\|'$ is a norm on $\mathbf{R}^2$. An open ball centered at the origin for $\|\cdot\|'$ is an open square (no boundary) centered at the origin with sides parallel to the axes. See the figure below. More generally, open balls for $\|\cdot\|'$ are open squares in $\mathbf{R}^2$ with sides parallel to the axes.



Unlike in Example 1.1, open balls for $\|\cdot\|_2$ and $\|\cdot\|'$ are not the same, but each is an open subset for the other norm:

- In each open ball for $\|\cdot\|'$, which is an ordinary open square, every point is contained in a Euclidean open ball that is inside the square.
- In a Euclidean open ball, each point is contained in an open square with sides parallel to the axes that is inside the Euclidean ball.

The picture below illustrates these two scenarios.



Thus the open subsets (not open balls!) of $\mathbf{R}^2$ for $\|\cdot\|_2$ and $\|\cdot\|'$ coincide.

Definition 1.3. Two norms on a K-vector space V are called *equivalent* if they define the same topology on V, or equivalently they define the same open subsets of V.

Our goal is two-fold: (i) describe equivalence of norms by a criterion analogous to the formula $|\cdot|' = |\cdot|^t$ for $t > 0$ linking equivalent absolute values $|\cdot|$ and $|\cdot|'$ on a field[1] and (ii) show all norms on a finite-dimensional vector space over a complete valued field[2] are equivalent. The analogue of $|\cdot|' = |\cdot|^t$ for norms will not be an equation, but inequalities.

2. Equivalent norms

Theorem 2.1. *Let $(K, |\cdot|)$ be a nontrivially valued field and V be a K-vector space. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on V are equivalent if and only if there are constants $A > 0$ and $B > 0$ such that $A\|v\|' \leq \|v\| \leq B\|v\|'$ for all $v \in V$.*

We can rewrite the inequalities in Theorem 2.1 in other ways:

- $(1/B)\|v\| \leq \|v\|' \leq (1/A)\|v\|$ for all $v \in V$ (this exchanges $\|\cdot\|$ and $\|\cdot\|'$)
- $\|v\|' \leq C\|v\|$ (for $C = 1/A$) and $\|v\| \leq C'\|v\|'$ (for $C' = B$) for all $v \in V$.

Proof. The theorem is obvious if $V = \{0\}$, so assume V is not $\{0\}$.

($\Longleftarrow$) First assume there are positive A and B such that $A\|v\|' \leq \|v\| \leq B\|v\|'$ for all $v \in V$. Then for each open set $U \subset V$ with respect to $\|\cdot\|$ and $v \in U$, there is an $\varepsilon > 0$ such that the open ε-ball with respect to $\|\cdot\|$ centered at v is contained in U:

$$\{w \in V : \|w - v\| < \varepsilon\} \subset U.$$

Since

$$\|w - v\|' < \frac{\varepsilon}{B} \Longrightarrow \|w - v\| < \varepsilon,$$

each open $\|\cdot\|$-ball centered at v contains an open $\|\cdot\|'$-ball centered at v, so U is open with respect to $\|\cdot\|'$. The proof that each open subset of V with respect to $\|\cdot\|'$ is open with respect to $\|\cdot\|$ is similar, using $\|v\|' \leq (1/A)\|v\|$ in place of $\|v\| \leq B\|v\|'$. Note this direction did not need $|\cdot|$ on K to be nontrivial.

($\Longrightarrow$) Assume $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Then the open unit ball centered at the origin in V relative to $\|\cdot\|$ is open relative to $\|\cdot\|'$ and the open unit ball centered at the origin in V relative to $\|\cdot\|'$ is open relative to $\|\cdot\|$, so there are $r > 0$ and $s > 0$ such that

$$\{v \in V : \|v\|' < r\} \subset \{v \in V : \|v\| < 1\}, \quad \{v \in V : \|v\| < s\} \subset \{v \in V : \|v\|' < 1\}. \tag{2.1}$$

[1] See https://kconrad.math.uconn.edu/blurbs/gradnumthy/equivabsvalues.pdf.

[2] A valued field is a field along with a choice of absolute value on it. For example, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ with their usual absolute values are valued fields for which $\mathbf{R}$ and $\mathbf{C}$ are complete while $\mathbf{Q}$ is not. If you have not studied valued fields besides $\mathbf{R}$ and $\mathbf{C}$, read all results here with those examples in mind. A trivially valued field is a field given the trivial absolute value, and it is complete since Cauchy sequences with respect to the trivial absolute value are eventually constant. For example, $\mathbf{Q}$ is complete for the trivial absolute value.

Since $|\cdot|$ is nontrivial on K, there is $\gamma \in K$ such that $|\gamma| > 1$. Then $|\gamma|^n \to \infty$ as $n \to \infty$ and $|\gamma|^n \to 0$ as $n \to -\infty$, so every positive real number is between two successive powers of $|\gamma|$ or equal to one of these powers. Therefore for each nonzero $v \in V$ there is an integer n such that

$$|\gamma|^n \leq \frac{1}{s}\|v\| < |\gamma|^{n+1}. \tag{2.2}$$

Then

$$\left\|\frac{1}{\gamma^{n+1}}v\right\| = \frac{1}{|\gamma|^{n+1}}\|v\| < s,$$

so $\|v/\gamma^{n+1}\|' < 1$ by (2.1). Thus $\|v\|' < |\gamma|^{n+1} = |\gamma||\gamma|^n \leq (|\gamma|/s)\|v\|$. Setting $B = |\gamma|/s$ we have $\|v\|' < B\|v\|$ for all nonzero $v \in V$, so $\|v\|' \leq B\|v\|$ for all v, including $v = 0$.

In a similar way, switching the roles of $\|\cdot\|$ and $\|\cdot\|'$ and using r in place of s, we get $\|v\| \leq (|\gamma|/r)\|v\|'$ for all $v \in V$, so $A\|v\| \leq \|v\|'$ where $A = r/|\gamma|$. □

Remark 2.2. The set of inequalities in the conclusion of Theorem 2.1 is often taken to be the definition of equivalent vector space norms, and inducing the same topology is then presented as a property of that definition rather than it being equivalent to that definition.

Theorem 2.1 is about norms on a vector space over a nontrivially valued field. Could the theorem have a counterexample when K is equipped with the trivial absolute value? Not if $\dim_K(V)$ is finite (see Theorem 3.4, which is true for trivially valued fields since they are complete), but here is a counterexample for an infinite-dimensional vector space.

Example 2.3. Let K be a field with the trivial absolute value and $V = \{(c_0, c_1, c_2, \ldots) : c_i \in K\}$ be the K-vector space of all sequences in K. For $0 < \varepsilon < 1$, define the ε-norm on V as follows:

$$\|(c_0, c_1, c_2, \ldots)\|_\varepsilon := \begin{cases} 0, & \text{if all } c_i = 0, \\ \varepsilon^n, & \text{if } c_n \neq 0, c_i = 0 \text{ for } i < n. \end{cases}$$

It is left to the reader to check this is a K-vector space norm on V.[3] (That K is trivially valued is relevant because by definition $\|cv\|_\varepsilon = \|v\|_\varepsilon$ for all $c \in K^\times$ and $v \in V$, and $|c|\,\|v\|_\varepsilon = \|v\|_\varepsilon$ when $c \in K^\times$ and K is trivially valued.)

The topology put on V by $\|\cdot\|_\varepsilon$ does not depend on ε: if ε and δ are both in $(0,1)$, then $\delta = \varepsilon^t$ for some $t > 0$ and a short calculation shows $\|v\|_\delta = \|v\|_\varepsilon^t$ for all $v \in V$, so $\|v\|_\delta < r \Leftrightarrow \|v\|_\varepsilon < r^{1/t}$. Thus open balls in V centered at 0 in one norm are open balls centered at 0 in the other norm, and by translation the same is true for open balls centered at each point in V.

For distinct ε and δ in $(0,1)$, the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\delta$ define the same topology on V, so they are equivalent norms in the sense of Definition 1.3, but it is *not* true that there are $A, B > 0$ such that $A\|v\|_\varepsilon \leq \|v\|_\delta \leq B\|v\|_\varepsilon$ for all $v \in V$. Indeed, suppose there were such A and B. For $n \geq 0$ let $v_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ have 1 in component n and 0 in all other components, so when $v = v_n$ the inequalities $A\|v\|_\varepsilon \leq \|v\|_\delta \leq B\|v\|_\varepsilon$ imply $A\varepsilon^n \leq \delta^n \leq B\varepsilon^n$. Therefore $A \leq (\delta/\varepsilon)^n \leq B$, and this holds for all $n \geq 0$. Thus the powers $(\delta/\varepsilon)^n$ are bounded away from both 0 and ∞, which is false: if $\delta < \varepsilon$ then $(\delta/\varepsilon)^n \to 0$ as $n \to \infty$, while if $\delta > \varepsilon$ then $(\delta/\varepsilon)^n \to \infty$ as $n \to \infty$.

[3] If we identify the formal power series ring $K[[X]]$ with V by $\sum_{n\geq 0} c_n X^n \leftrightarrow (c_0, c_1, c_2, \ldots)$, then the ε-norm on V corresponds to the X-adic norm on $K[[X]]$ where $|X| = \varepsilon$.

3. Norms on finite-dimensional spaces

We are interested in comparing norms on each finite-dimensional vector space over a complete valued field. First we will look at real vector spaces since they are traditionally handled by a method that is not directly applicable to the more general case.

Theorem 3.1. *All norms on a finite-dimensional real vector space are equivalent.*

Proof. Let V be a finite-dimensional real vector space. The theorem is obvious if $V = \{0\}$, so now suppose $n := \dim_{\mathbf{R}}(V) \geq 1$. Pick a basis $\{e_1, \ldots, e_n\}$ of V. Define

$$\left\| \sum_{i=1}^n c_i e_i \right\|_V := \|(c_1, \ldots, c_n)\|_2 = \sqrt{\sum_{i=1}^n c_i^2}, \tag{3.1}$$

so $\|\cdot\|_V$ is a norm on V since the Euclidean norm $\|\cdot\|_2$ is a norm on $\mathbf{R}^n$. The map $\mathbf{R}^n \to V$ where $(c_1, \ldots, c_n) \mapsto \sum_{i=1}^n c_i e_i$ is an isometry for the metrics on $\mathbf{R}^n$ and V associated to $\|\cdot\|_2$ and $\|\cdot\|_V$.

For each norm $\|\cdot\|$ on V we will find $A > 0$ and $B > 0$ such that $A\|v\|_V \leq \|v\| \leq B\|v\|_V$ for all $v \in V$, so every norm on V is equivalent to $\|\cdot\|_V$ and thus all norms on V are equivalent to each other since norm-equivalence is an equivalence relation. The existence of B will be explicit, while the existence of A will rely on a compactness argument.

Step 1: There is $B > 0$ such that $\|v\| \leq B\|v\|_V$ for all $v \in V$.

For $v = \sum_{i=1}^n c_i e_i$ in V,

$$\|v\| = \left\| \sum_{i=1}^n c_i e_i \right\| \leq \sum_{i=1}^n |c_i| \, \|e_i\| \leq \sqrt{\left(\sum_{i=1}^n c_i^2 \right) \left(\sum_{i=1}^n \|e_i\|^2 \right)},$$

where the second inequality is the Cauchy–Schwartz inequality on n-tuples of real numbers. The number on the right is $\|v\|_V \sqrt{\sum_{i=1}^n \|e_i\|^2} = B\|v\|_V$, where $B = \sqrt{\sum_{i=1}^n \|e_i\|^2} > 0$. Thus $\|v\| \leq B\|v\|_V$ for all $v \in V$.

Step 2: The norm $\|\cdot\| : V \to \mathbf{R}$ is continuous for the topology on V coming from $\|\cdot\|_V$. In fact, $\|\cdot\|$ is uniformly continuous for that topology on V.

For $\varepsilon > 0$ we seek $\delta > 0$ such that for all v and w in V, $\|w - v\|_V < \delta \Longrightarrow |\,\|w\| - \|v\|\,| < \varepsilon$. The triangle inequality for $\|\cdot\|$ implies[4] $|\,\|w\| - \|v\|\,| \leq \|w - v\|$, so it suffices to find $\delta > 0$ such that $\|w - v\|_V < \delta \Longrightarrow \|w - v\| < \varepsilon$. By Step 1, $\|w - v\| \leq B\|w - v\|_V < B\delta$, so let $\delta = \varepsilon/B$, which is independent of v and w. Then $\|w - v\|_V < \delta \Longrightarrow \|w - v\| < B\delta = \varepsilon$.

Step 3: There is $A > 0$ such that $A\|v\|_V \leq \|v\|$ for all $v \in V$.

The case $v = 0$ is easy, so focus on nonzero v. Let S be the unit sphere in V relative to $\|\cdot\|_V$:

$$S = \{v \in V : \|v\|_V = 1\} = \left\{ \sum_{i=1}^n c_i e_i : c_1^2 + \cdots + c_n^2 = 1 \right\}.$$

This is compact in the topology on V induced by $\|\cdot\|_V$ since the unit sphere in $\mathbf{R}^n$ is compact.

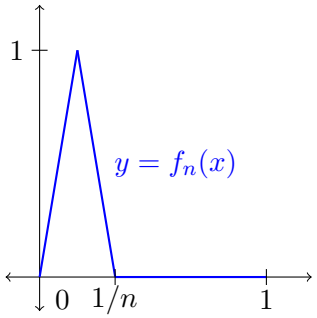
Using the topology on V induced by $\|\cdot\|_V$, $\|\cdot\| : V \to \mathbf{R}$ is continuous by Step 2 and $\|\cdot\|$ is positive on S since $0 \notin S$. By compactness of S, $\|\cdot\|$ has a positive lower bound on S. Call it A, so $\|v\|_V = 1 \Longrightarrow \|v\| \geq A$. For nonzero v, set $c = \|v\|_V > 0$, so $\|v/c\|_V = \|v\|_V / c = 1$. Then $v/c \in S$, so $\|v/c\| \geq A$. Thus $\|v\| \geq Ac = A\|v\|_V$. □

[4] We have $\|w\| \leq \|w - v\| + \|v\|$, so $\|w\| - \|v\| \leq \|w - v\|$. Likewise $\|v\| - \|w\| \leq \|v - w\| = \|w - v\|$. Thus $|\,\|w\| - \|v\|\,| \leq \|w - v\|$.

Remark 3.2. Theorem 3.1 has counterexamples among infinite-dimensional spaces. For example, let $V = C([0,1])$ be the vector space of continuous functions $f\colon [0,1] \to \mathbf{R}$. Define two norms on $C([0,1])$:

$$\|f\| = \max_{0 \le x \le 1} |f(x)|, \quad \|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Clearly $\|f\|_1 \le \|f\|$ for all f, but there is no $B > 0$ such that $\|f\| \le B\,\|f\|_1$ for all f. Indeed, let $f_n\colon [0,1] \to \mathbf{R}$ be 0 outside of $[0,1/n]$ and have a triangular shape over $[0,1/n]$ with a peak at height 1, as in the picture below.



Then $\|f_n\| = 1$ and $\|f_n\|_1 = 1/(2n)$. If there were $B > 0$ such that $\|\cdot\| \le B\,\|\cdot\|_1$ on $C([0,1])$, then taking $f = f_n$ gives us $1 \le B/(2n)$ for all $n \ge 1$, which is impossible. (Intuitively, the reason $\|\cdot\|$ and $\|\cdot\|_1$ are not equivalent is that $f_n \to 0$ in the norm $\|\cdot\|_1$ but $f_n \not\to 0$ in the norm $\|\cdot\|$.)

Now we look at finite-dimensional vector spaces V over any field K that is complete for an absolute value. If K is locally compact for its absolute value, then there is a norm on V whose unit sphere is compact and the proof of Theorem 3.1 can be adapted to norms on V, but some valued fields K are *complete without being locally compact* (the simplest example of that is the p-adic complex numbers $\mathbf{C}_p$). We can't adapt the proof of Theorem 3.1 to norms on finite-dimensional vector spaces over a valued field K that is complete but not locally compact. Another method is needed.

In place of the norm (3.1) when $K = \mathbf{R}$, we'll use the following construction of a norm on V that depends on a basis in a different way.

Lemma 3.3. *Let V be a nonzero finite-dimensional vector space over a valued field $(K, |\cdot|)$. For a basis $\{e_1, \ldots, e_n\}$ of V over K, set*

$$\left\| \sum_{i=1}^n c_i e_i \right\|_\infty := \max_{1 \le i \le n} |c_i| \tag{3.2}$$

where $c_i \in K$. This is a norm on V, and if K is complete with respect to $|\cdot|$ then V is complete with respect to $\|\cdot\|_\infty$.

Proof. The axioms of a vector space norm are easy to verify for $\|\cdot\|_\infty$, *e.g.*, the triangle inequality for $\|\cdot\|_\infty$ follows from the triangle inequality for $|\cdot|$ on basis coefficients.

Suppose K is complete with respect to $|\cdot|$. To prove V is complete with respect to $\|\cdot\|_\infty$, let $\{v_m\}$ be a Cauchy sequence in V with respect to $\|\cdot\|_\infty$: for each $\varepsilon > 0$ there is an $M \ge 1$ such that $\ell, m \ge M \Longrightarrow \|v_\ell - v_m\|_\infty < \varepsilon$. Writing v_m in terms of the chosen basis of V as

$$v_m = c_{1m}e_1 + c_{2m}e_2 + \cdots + c_{nm}e_n,$$

the definition of the norm $\|\cdot\|_\infty$ tells us

$$\ell, m \geq M \Longrightarrow |c_{i\ell} - c_{im}| < \varepsilon$$

for each i from 1 to n, so each sequence $\{c_{im}\}_{m\geq 1}$ in K is Cauchy with respect to $|\cdot|$. Since K is complete with respect to $|\cdot|$, there is a limit: $\lim_{m\to\infty} c_{im} = c_i$ in $(K, |\cdot|)$. Then $v = \sum c_i e_i$ is a limit for the v_m's in V:

$$\|v - v_m\|_\infty = \max_{1\leq i\leq n} |c_i - c_{im}| \to 0 \text{ as } m \to \infty. \qquad \square$$

Theorem 3.4. *All norms on a finite-dimensional vector space over a complete valued field are equivalent.*

Proof. Let $(K, |\cdot|)$ be a complete valued field and V be a finite-dimensional K-vector space. To prove all norms on V are equivalent, the case $V = \{0\}$ is trivial, so assume $n := \dim_K(V) \geq 1$. We will prove the theorem by induction on n using the argument from [3, pp. 132–133]. (I first learned a proof of this theorem with a different induction proof in [1, pp. 116–117], which is also presented in [2, pp. 173–176].)

Pick a K-basis $\{e_1, \ldots, e_n\}$ of V and define $\|\cdot\|_\infty$ from this basis by (3.2). For an arbitrary norm $\|\cdot\|$ on V, we will find $A > 0$ and $B > 0$ such that $A\|v\|_\infty \leq \|v\| \leq B\|v\|_\infty$ for all $v \in V$; thus each norm on V is equivalent to $\|\cdot\|_\infty$, so all norms on V are equivalent to each other. As in the proof of Theorem 3.1, it will be easier to find B than A.

Step 1: There is $B > 0$ such that $\|v\| \leq B\|v\|_\infty$ for all $v \in V$.

For $v = \sum_{i=1}^n c_i e_i$ in V,

$$\|v\| = \left\|\sum_{i=1}^n c_i e_i\right\| \leq \sum_{i=1}^n |c_i|\,\|e_i\| \leq (\max|c_i|)\sum_{i=1}^n \|e_i\|,$$

so $\|v\| \leq B\|v\|_\infty$ where $B = \sum_{i=1}^n \|e_i\| > 0$.

Step 2: There is $A > 0$ such that $A\|v\|_\infty \leq \|v\|$ for all $v \in V$.

If $n = 1$, so $V = Ke_1$, write $v = c_1 e_1$. Then $\|v\| = |c_1|\,\|e_1\| = A\|v\|_\infty$ where $A = \|e_1\|$.

Assume $n \geq 2$ and that Step 2 is proved for all K-vector spaces of dimension $n-1$. For $1 \leq i \leq n$, set $V_i := \sum_{j\neq i} Ke_j$, so $\dim_K(V_i) = n-1$ and thus all norms on V_i are equivalent. In particular, the restrictions of $\|\cdot\|$ and $\|\cdot\|_\infty$ to norms on V_i are equivalent, so V_i is complete with respect to the restriction of $\|\cdot\|$ by Lemma 3.3. A complete subset of a metric space is closed, so V_i is closed in V when V has the topology from $\|\cdot\|$. Then $V_i + e_i$ (the vectors in V with e_i-coefficient 1) is closed in V, so the finite union $C := \bigcup_{i=1}^n (V_i + e_i)$ is closed in V. Obviously $0 \notin C$. The complement of C in V is open, so there's an $A > 0$ such that the open ball $O := \{v \in V : \|v\| < A\}$ is in the complement of C.

We'll show $A\|v\|_\infty \leq \|v\|$ for all $v \in V$. This is easy when $v = 0$, so let $v \neq 0$. Write $v = c_1 e_1 + \cdots + c_n e_n$ and $\|v\|_\infty = |c_k| > 0$, so $v/c_k \in V_k + e_k$. Thus $v/c_k \notin O$, so $\|v/c_k\| \geq A$, which implies $\|v\| \geq A|c_k| = A\|v\|_\infty$. $\square$

This proof shows two norms are bounded by constant multiples of each other, so it does not rely on the direction of Theorem 2.1 that needs the absolute value on K to be nontrivial. That is why Theorem 3.4 is valid when K is a trivially valued field since it is complete.

Corollary 3.5. *Let $(K, |\cdot|)$ be a complete valued field and $(V, \|\cdot\|)$ be a finite-dimensional normed K-vector space. Every subspace of V is closed with respect to $\|\cdot\|$.*

Proof. This is obvious when $V = \{0\}$, so assume $V \neq \{0\}$. Let $n = \dim_K(V) \geq 1$.

Let W be a subspace of V. To say W is closed with respect to $\|\cdot\|$ is unchanged if $\|\cdot\|$ is replaced by an equivalent norm on V (using an equivalent norm doesn't change the meaning of an open subset of V, so by passing to complements it doesn't change the meaning of a closed subset of V). By Theorem 3.4, all norms on V are equivalent, so it suffices to prove the theorem using a norm on V that is adapted to the choice of subspace W.

Set $d = \dim_K(W)$. That W is closed in V is obvious if $d = 0$, since then $W = \{0\}$ and a one-point subset of a normed vector space — or of a metric space – is closed (the complement is open). The result is also obvious if $d = n$, since then $W = V$ and V is a closed subset of V. We now suppose $n \geq 2$ and $1 \leq d \leq n-1$.

Pick a basis $\{e_1, \ldots, e_d\}$ of W and extend it to a basis $\{e_1, \ldots, e_d, e_{d+1}, \ldots, e_n\}$ of V. Let $\|\cdot\|_\infty$ be the norm on V defined from this basis by (3.2). Then

$$W = \left\{ \sum_{i=1}^{n} c_i e_i : c_{d+1} = \cdots = c_n = 0 \right\}$$

and $\|\cdot\|$ on V is equivalent to $\|\cdot\|_\infty$ by Theorem 3.4, so it suffices to show W is closed with respect to $\|\cdot\|_\infty$. If $\{w_m\} \subset W$ and $w_m \to v$ in V for the topology of $\|\cdot\|_\infty$, write $v = a_1 e_1 + \cdots + a_n e_n$ with $a_i \in K$. Since the e_i-coefficient of w_m is 0 for $i > d$ by the definition of $\|\cdot\|_\infty$, for $i > d$ we have

$$|a_i| = |0 - a_i| \leq \|w_m - v\|_\infty .$$

Since $\|w_m - v\|_\infty \to 0$ as $m \to \infty$ we get $|a_i| = 0$, so $a_i = 0$, when $i > d$. Thus $v \in W$. □

Appendix A. Equivalence in terms of convergence

We defined equivalence of two norms on a vector space V in terms of them defining the same open subsets of V. Equivalence of norms can also be characterized in terms of having the same convergent sequences and limits.

Theorem A.1. *Let $(K, |\cdot|)$ be a valued field. The following properties of norms $\|\cdot\|$ and $\|\cdot\|'$ on a K-vector space V are equivalent to each other.*

1. *The open subsets of V defined by $\|\cdot\|$ and $\|\cdot\|'$ are the same.*
2. *The convergent sequences and their limits in V defined by $\|\cdot\|$ and $\|\cdot\|'$ are the same.*

A more general result applies to two metrics on a set, not just two norms on a vector space.

Theorem A.2. *The following properties of metrics d and d' on a set X are equivalent to each other.*

1. *The open subsets of X defined by d and d' are the same.*
2. *The convergent sequences and their limits in X defined by d and d' are the same.*

When $X = V$ is a vector space and the two chosen metrics on V are induced by norms on V, then Theorem A.2 becomes Theorem A.1.

Proof. (1) $\Longrightarrow$ (2): Convergence in X relative to a metric can be described in terms of open subsets of X for that metric: to say $d(x_n, x) \to 0$ as $n \to \infty$ means for every open subset $U \subset X$ relative to d such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. By (1), a subset of X that is open relative to d' is also dopen relative to d, so for every open subset $U \subset X$ relative to d' such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. Thus $x_n \to x$ relative to the topology defined by d', which means $d'(x_n, x) \to 0$. Since (1)

is symmetric for d and d', a similar argument shows each convergent sequence in X relative to d' is convergent relative to d with the same limit.

(2) $\Longrightarrow$ (1): Since open subsets are complements of closed subsets, proving (1) is equivalent to proving the closed subsets of X defined by d and d' are the same. Let $C \subset X$ be closed relative to d. To show C is closed relative to d', let $\{x_n\}$ be a sequence in C that converges to some $x \in X$ relative to d': $d'(x_n, x) \to 0$. By hypothesis, $d(x_n, x) \to 0$, so $x \in C$ since C is closed relative to d. Hence every sequence in C that converges in X relative to d' has its d'-limit in C, so C is closed relative to d'.

The proof that every closed subset of X relative to d' is closed relative to d is the same, with the roles of d and d' exchanged. □

Theorem A.2 does not directly generalize to topological spaces: different topologies on a set, even when both are Hausdorff, can have the same convergent sequences and limits.

Example A.3. A Banach space X has two standard Hausdorff topologies: the norm and weak topologies. Basic weak-open sets around 0 are $U(f_1, \ldots, f_k, \varepsilon) := \{x \in X : |f_i(x)| < \varepsilon\}$ for finite subsets $\{f_1, \ldots, f_k\}$ in X^*. Other basic weak-open sets are translates of these. For infinite-dimensional X we'll show these two topologies on X are different and give an infinite-dimensional X where these topologies have the same convergent sequences.

Step 1: Weak-open sets are norm-open. It suffices (why?) to show each $U(f_1, \ldots, f_k, \varepsilon)$ is norm-open. It equals $\bigcap_{i=1}^n {f_i}^{-1}((-\varepsilon, \varepsilon))$, which is a finite intersection of norm-open sets.

Step 2: Some norm-open sets are not weak-open, so the norm and weak topologies are different. Each $U(f_1, \ldots, f_k, \varepsilon)$ contains $\bigcap_{i=1}^n \ker(f_i)$, which has finite codimension in X, so $\bigcap_{i=1}^n \ker(f_i) \neq \{0\}$ since X is infinite-dimensional. Let x be nonzero in $\bigcap_{i=1}^n \ker(f_i)$, so the intersection contains $\mathbf{R}x$ and thus is not norm-bounded, so $U(f_1, \ldots, f_k, \varepsilon)$ and its translates are never in an open ball in the norm topology on X. Thus open balls in the norm topology on X are not weak-open. By this and Step 1, the weak-topology has fewer open sets than the norm-topology (that is why the weak topology is called "weak").

Step 3: Find an infinite-dimensional X in which weak-convergent and norm-convergent sequences are the same with the same limits. Step 1 implies a norm-convergent sequence in any Banach space is weak-convergent with the same limit[5]. What is an X where weak-convergence implies norm-convergence with the same limit? That never happens for infinite-dimensional reflexive spaces, like L^p-spaces for $1 < p < \infty$. It also fails in $L^1([0,1])$, where the sequence $x_n(t) = e^{i\pi nt}$ converges weakly to 0 (by the Riemann–Lebesgue lemma, since $L^\infty([0,1]) \subset L^1([0,1])$) but it does not converge in norm since $|x_m - x_n|_1 = 4/\pi$ if $m \neq n$. A valid X is ℓ^1: Schur proved that in ℓ^1, weak-convergence implies norm-convergence with the same limit. See https://math.stackexchange.com/questions/42609 for some proofs, which are tricky or technical.

Thus *sequences* may not capture everything about a topology. If we use nets, which are a generalization of sequences, then we get a nicer situation: a topology is determined by its convergent nets and their limits. See https://mathoverflow.net/questions/19285.

References

[1] J. W. S. Cassels, "Local Fields," Cambridge Univ. Press, Cambridge, 1986.
[2] F. Q. Gouvea, "p-adic Numbers: An Introduction," 3rd edition, Springer-Verlag, 2020.
[3] J. Neukirch, "Algebraic Number Theory," Springer-Verlag, Berlin, 1999.

[5] If $\|x_n - x\| \to 0$ and U is a weak-open neighborhood of x, then Step 1 tells us some open ball B centered at x is in U, so $x_n \in B$ for $n \gg 0$ by norm-convergence of $\{x_n\}$ to x. Thus $x_n \in U$ for $n \gg 0$.