

Essays on Derivatives Pricing

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1 Introduction

The field of quantitative finance has been criticized in the mainstream media lately and been accused of being one of the causes of the financial crisis. Convenient as this explanation may be, my belief is that a part of the solution to the crisis is to use more (and not less) sophisticated financial models, and most importantly, to be aware of their limitations.

Following that belief, this thesis consists of three independent and self-contained papers, all dealing with topics in derivatives pricing. Detailed summaries of the papers are provided in the next section, while a brief motivation for each paper is given here.

The first paper considers the pricing of traffic light options, which are appropriate instruments for life and pension companies to hedge against simultaneous downward movements in interest rate levels and stock markets. In these scenarios, life and pension companies are exposed for two reasons. First, the duration is typically much longer on the liability side than on the asset side, exposing the company to negative shocks to interest rate levels. Second, many life and pension companies have issued guarantees on policy holder contributions, which, with the low interest rate levels today, forces the companies to invest in the stock market in order to capture the higher expected return here. This investment behavior exposes the companies to negative shocks of the stock market as well.

The second paper is on the topic of credit risk, which is at the very core of the crisis. The estimated total notional value of the credit default swap market in the US rose from \$900 billion in 2000 to the breathtaking number of more than \$60 trillion in the beginning of 2008 – roughly twice the size of the entire US stock market value. As a consequence of the financial crisis, the market moved in the opposite direction through 2008 and 2009 and was down to a total notional value of \$30 trillion by March 2009. This development has been followed by a tremendous growth in the academic literature on credit risk, with a tendency in recent years to focus on multi-name default modeling. Partly motivated by the ongoing financial turmoil which has almost entirely wiped out the market for multivariate credit instruments, we take a step back and focus on single-name default modeling and introduce two new model classes for modeling of the default time of a company.

Finally, in the third paper we propose a consistent pricing model for index

and volatility derivatives. During the crisis, volatility levels have been fluctuating a lot, with the VIX volatility index reaching as much as 80% in October 2008, and modeling of variance is getting increasingly important. Furthermore, with the existence of a liquid market for derivatives with variance as underlying, such as VIX options, VIX futures and a well-developed over-the-counter market for options on variance swaps, it is important to consider models that are able to fit these markets while consistently pricing vanilla options on the underlying index.

2 Summary of Essays

2.1 English Summaries

Pricing of Traffic Light Options and other Hybrid Products

Published in International Journal of Theoretical and Applied Finance, vol. 12, no. 5, 2009.

This paper considers derivatives with payoffs that depend on a stock index and underlying LIBOR rates. A *traffic light option* pricing formula is derived under log-normality assumptions on the underlying processes. The traffic light option is aimed at the Danish life and pension sector to help companies stay solvent in the traffic light stress test system introduced by the Danish Financial Supervisory Authorities in 2001. Similar systems are now being implemented in several other European countries.

A pricing approach for general payoffs is presented and illustrated with simulation via the pricing of a hybrid derivative known as the EUR Sage Note. The approach can be used to price many existing structured products.

Sato Processes in Default Modeling

Joint work with Elisa Nicolato. Forthcoming in Applied Mathematical Finance.

In reduced form default models, the instantaneous default intensity is the classical modeling object. Survival probabilities are then given by the Laplace transform of the cumulative hazard defined as the integrated intensity process. Instead, recent literature tends to specify the cumulative hazard process directly.

Within this framework we present a new model class where cumulative hazards are described by self-similar additive processes, also known as Sato processes. Furthermore, we analyze specifications obtained via a simple deterministic time-change of a homogeneous Lévy process. While the processes in these two classes share the same average behavior over time, the associated intensities exhibit very different properties.

Concrete specifications are calibrated to data on all the single names included in the iTraxx Europe index. The performances are compared with those of the classical CIR intensity and a recently proposed class of intensity models based on Ornstein-Uhlenbeck type processes. It is shown that the time-inhomogeneous Lévy models achieve comparable calibration errors with fewer parameters, and with more stable parameter estimates over time. However, the calibration performance of the Sato processes and the time-change specifications are practically indistinguishable.

A Consistent Pricing Model for Index Options and Volatility Derivatives

Joint work with Rama Cont.

We propose and study a flexible modeling framework for the joint dynamics of an index and a set of forward variance swap rates written on this index, allowing volatility derivatives and options on the underlying index to be priced consistently. Our model reproduces various empirically observed properties of variance swap dynamics and allows for jumps in volatility and returns.

An affine specification using Lévy processes as building blocks leads to analytically tractable pricing formulas for options on the VIX as well as efficient numerical methods for pricing of European options on the underlying asset. The model has the convenient feature of decoupling the vanilla skews from spot/volatility correlations and allowing for different conditional correlations in large and small spot/volatility moves.

We show that our model can simultaneously fit prices of European options on S&P 500 across strikes and maturities as well as options on the VIX volatility index. The calibration of the model is done in two steps, first by matching VIX option prices and then by matching prices of options on the underlying.

2.2 Danish Summaries

Pricing of Traffic Light Options and other Hybrid Products

Publiceret i International Journal of Theoretical and Applied Finance, vol. 12, no. 5, 2009.

Denne artikel omhandler derivater med betalinger, der er betingede af et aktieindeks og underliggende LIBOR-renter. En formel til prisfastsættelse af *traffic light*-optioner er udledt under log-normale antagelser for de underliggende processer. *Traffic light*-optionen er rettet mod den danske liv og pensionssektor for at hjælpe virksomheder forblive solvente i trafiklysens system til stresstest, der blev indført af Finanstilsynet i 2001. Lignende systemer er nu ved at blive implementeret i flere andre europæiske lande.

En fremgangsmåde til prisfastsættelse af derivater med generelle betalinger præsenteres og illustreres med simulering via prissætningen af et hybridaktiv betegnet som *EUR Sage Note*. Denne fremgangsmåde kan anvendes til at prisfastsætte mange eksisterende strukturerede produkter.

Sato Processes in Default Modeling

Skrevet sammen med Elisa Nicolato. Udkommer i Applied Mathematical Finance.

Den instantane konkursintensitet er typisk modelleringsobjektet i reduceret form konkursmodeller. Overlevelsessandsynligheder er da givet ved Laplace-transformeringen af den kumulative risiko defineret som den integrerede intensitetsproces. Nyere litteratur har i stedet vist en tendens til at specificere den kumulative risikoprocess direkte.

Inden for disse rammer præsenterer vi en ny klasse af modeller, hvor kumulative risici bliver beskrevet som *self-similar additive* processer, også kendt som Sato-processer. Desuden analyserer vi specifikationer opnået ved en deterministisk tidsændring af en homogen Lévy-proces. Mens processer i disse to klasser har den samme gennemsnitlige adfærd over tid, udviser de tilhørende intensiteter meget forskellige egenskaber.

Konkrete specifikationer er kalibreret til data på alle de enkelte navne i *iTraxx Europe* indekset. Resultaterne sammenlignes med den klassiske CIR-intensitet og en nyligt foreslået klasse af intensitetsmodeller baseret på Ornstein-Uhlenbeck type processer. Det vises, at de tids-inhomogene Lévy-modeller

opnår sammenlignelige kalibreringsfejl med færre parametre, og over tid mere stabile parameterestimater. Imidlertid er kalibreringsresultaterne for Sato-processerne og de tidsændrede specifikationer næsten ikke til at skelne fra hinanden.

A Consistent Pricing Model for Index Options and Volatility Derivatives

Skrevet sammen med Rama Cont.

I denne artikel foreslår og studerer vi en fleksibel model for den fælles dynamik i et indeks og et sæt af fremtidige variansswap-rater defineret på dette indeks, så volatilitetsderivater og optioner på det underliggende indeks prisfastsættes konsistent. Modellen gengiver forskellige empiriske egenskaber for dynamikken i variansswaps og tillader spring i volatilitet og afkast.

Ved at bruge Lévy-processer som grundlag udledes analytisk håndterbare prisformler for optioner på VIX indekset samt effektive numeriske metoder til prisfastsættelse af europæiske optioner på det underliggende aktiv. Modellen har den praktiske egenskab, at den afkobler det implicitte volatilitetsskew og spot/volatilitet-korrelationen og samtidig giver plads til forskellige korrelationer i store og små spot/volatilitets-udsving.

Vi viser, at modellen simultant kan tilpasses til europæiske optionspriser på S&P 500 på tværs af forskellige *strikes* og løbetider samt optioner på VIX-volatilitetsindekset. Kalibreringen af modellen udføres trinvis. Først ved at tilpasse til VIX-optionspriser og derefter optioner på det underliggende indeks.

Pricing of Traffic Light Options and other Hybrid Products

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Abstract

This paper considers derivatives with payoffs that depend on a stock index and underlying LIBOR rates. A *traffic light option* pricing formula is derived under lognormality assumptions on the underlying processes. The traffic light option is aimed at the Danish life and pension sector to help companies stay solvent in the traffic light stress test system introduced by the Danish Financial Supervisory Authorities in 2001. Similar systems are now being implemented in several other European countries.

A pricing approach for general payoffs is presented and illustrated with simulation via the pricing of a hybrid derivative known as the EUR Sage Note. The approach can be used to price many existing structured products.

Key words: Traffic light option; LIBOR market model; correlation; simulation; hybrid products.

1 Introduction

Little attention has been directed towards the pricing of derivatives with payoffs depending on both a stock index and some benchmark interest rates, even though parts of the banking community are now offering their clients the possibility to invest in such hybrid products. The aim of this paper is to show how European style derivatives with payoffs, depending on both a stock index and underlying LIBOR rates, can be priced, and special attention will be paid to the pricing of *traffic light options* first introduced in Jørgensen (2007). The traffic light option is a tailor made derivative aimed at the Danish Life and Pension (L&P) sector to help companies stay solvent in the so-called traffic light scenarios introduced by the Danish Financial Supervisory Authorities (DFSA) in 2001. In short, the traffic light system is a supervision tool that consists of different scenarios, where both the interest rate level and stock prices fall simultaneously. L&P companies are typically exposed in these scenarios for two reasons. First, the duration is typically much longer on the liability side than on the asset side, exposing the company to negative shocks to interest rate levels. Second, many L&P companies have issued guarantees on policy holder contributions, which, with the low interest rate levels today, forces the companies to invest in the stock market in order to capture the higher expected return here. This investment behavior exposes the companies to negative shocks of the stock market. The company is labeled in red light if its base capital cannot stay above a certain critical level (4.5% of the pension obligations) in the scenario where interest rate levels decrease 70 bps, the stock market by 12%, and real estate by 8%. If this happens, the company is strictly monitored by the DFSA and forced to submit monthly solvency reports. The company is labeled in yellow light and required to submit quarterly solvency reports if it is able to stay above the critical level in the red scenario, but not in the yellow scenario, which consists of a 100 bps decrease in interest rates, 30% decrease in the stock market and a 12% decrease in real estate values. Finally, the company is "healthy" and labeled in green light if it stays above the level in the yellow scenario. In the above described scenarios, the traffic light option has a payoff.¹

¹One could introduce a "real estate" payoff in the traffic light option as well, but this is not done since the real estate part of total portfolio value is normally insignificant.

It is well known that when the stock market falls, many investors move their investments to "safe harbour" and invest in bonds. This drives the interest rates down, meaning that the option will be very valuable under these circumstances. A recent example where this happened was the Wall Street panic in the beginning of January 2008, where sharp drops in the US stock market drove investors to the bond market, and further led the FED to lower the federal funds rate from 4.25% to 3.50% on January 22nd and an extra 0.5%-point on Wednesday the following week. This is an example of a scenario where the traffic light option can help pension companies remain solvent.

The model in Jørgensen (2007) is an extension of the Black-Scholes-Vasicek model, where the benchmark interest rate is determined by the dynamics of the instantaneous short rate, and a closed form solution for the price of the traffic light option is derived. The use of an instantaneous short rate process for the development in interest rates has certain drawbacks. Especially, all zero-coupon bond yields in a one-factor model are perfectly dependent, making this setup less flexible. Basing a model on instantaneous rates further introduces problems from a practical perspective, since observed market rates are quoted on a discrete time basis. Among other things, this lack of flexibility led to the development of the LIBOR market model in 1997 by Brace, Gatarek, and Musiela (1997), Miltersen, Sandmann, and Sondermann (1997) and Jamshidian (1997). The LIBOR market model has been applied in several papers, but the pricing of hybrid products in a LIBOR market framework has not been studied. The market has adopted the LIBOR market model and it is now the preferred setup for many practical purposes.

This paper has two contributions: First, a closed form solution for the traffic light option price is derived, in the case where the underlying rate is some LIBOR rate. The closed form derivation is possible under lognormal distributional assumptions on the underlying processes. This closed form solution is new. Second, under more general assumptions and more advanced payoffs, one will be forced to use Monte Carlo simulation in order to price hybrid products. A pricing approach in these cases is presented and it is done under the spot LIBOR measure introduced in Jamshidian (1997). This is the equivalent martingale measure which corresponds to a discrete bank account as the numeraire. The method is applied to price a variant of a derivative issued by BNP Paribas called the EUR Sage Note, which has a payout similar to the ones seen in

various structured products. The market for structured products has exploded in recent years. The volume of new issuances of structured products in the US amounted to USD 28 billion in 2003 and have risen to over 100 billion dollars in 2007.² A major part of these structured products has a payment profile that depends on both the performance of a stock index and one or more interest rates.

In Section 2, we set up the model, and the price of the traffic light option is derived in the lognormal setting. A pricing approach for more general payoffs is also presented. Then numerical implementation of the closed form solution and with Monte Carlo simulation is performed in Section 3. The illustration of the Monte Carlo implementation is obtained via the pricing of the EUR Sage Note, but it is straightforward to apply the approach to other payoffs. Finally, Section 4 concludes.

2 Model

In this section, a framework for modeling derivatives with payoffs depending on both an underlying LIBOR rate and an underlying stock index is defined.

It is assumed that a set of settlement dates is given,

$$0 \leq T_0 < T_1 < \dots < T_n.$$

This is known as the tenor structure. Later, in the Monte Carlo implementation, it is assumed that the length between two tenor dates is fixed (i.e., $\tau_j = T_j - T_{j-1} = \tau$) but in general $\tau_j \neq \tau_i$. The existence of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the physical probability measure $\mathbb{P}$ is assumed. It is also assumed that at time zero and for each tenor date T_j , the price of a zero coupon bond maturing at that date $B(0, T_j)$ is given. And finally, all the standard assumptions of efficient and perfect markets are assumed. We start by introducing the LIBOR rate $L_i(t)$ defined by

$$\begin{aligned} 1 + \tau_{i+1} L_i(t) &= \frac{B(t, T_i)}{B(t, T_{i+1})} \\ L_i(t) &= \frac{1}{\tau_{i+1}} \left(\frac{B(t, T_i)}{B(t, T_{i+1})} - 1 \right). \end{aligned} \tag{1}$$

²Source: Structured Products Association.

$L_i(t)$ is the prevailing simply compounded forward rate from T_i to T_{i+1} as seen at time t . Let $\mathbb{Q}^T$ denote the equivalent martingale measure (EMM) corresponding to the numeraire $B(t, T)$. Then it is seen from (1) that the process L_i is a martingale under the forward measure $\mathbb{Q}^{T_{i+1}} = \mathbb{Q}^{i+1}$ due to the Fundamental Theorem of Asset Pricing (FTAP) in Delbaen and Schachermayer (1998). Assuming a continuous evolution gives rise to the LIBOR rate dynamics

$$dL_i(t) = \lambda(t, T_i) dW^{i+1}(t),$$

where $\lambda(t, T_i)$ is some (multivariate) stochastic process and W^{i+1} is a (multivariate) Wiener process under $\mathbb{Q}^{i+1}$. In the lognormal LIBOR market model, the diffusion term is given by $\lambda(t, T_i) = L_i(t) \lambda_i(t)$ for some deterministic function $\lambda_i(t)$ such that the dynamics take the form

$$dL_i(t) = L_i(t) \lambda_i(t) dW^{i+1}(t). \quad (2)$$

In this setup, the pricing of a T_i -claim $C(S(T_0), \dots, S(T_i), L_0(T_0), \dots, L_n(T_i))$ is considered, where S is some stock index and $i \leq n$. We know from the FTAP that under the assumption of no arbitrage possibilities, an EMM $\mathbb{Q}^N$ exists, so that the price is given by

$$\pi(t) = N(t) \mathbb{E}^{\mathbb{Q}^N} \left[\frac{1}{N(T_i)} C(S(T_0), \dots, S(T_i), L_0(T_0), \dots, L_n(T_i)) \mid \mathcal{F}_t \right], \quad (3)$$

where N is a positive price process. Here the question of what numeraire N to choose (or equivalently what EMM $\mathbb{Q}^N$) naturally arises. An obvious choice is to let the numeraire be the zero-coupon bond maturing at time T_{n+1} , since we know that L_n is a martingale under this measure with dynamics given by (2). With this choice the problem of finding the dynamics of S and all the other LIBOR rates under this measure remains. So when the payoff is more complex, it is convenient to choose another numeraire. One possibility is to use the discrete savings account, which is dealt with later in the paper. First, a price formula for the traffic light option is derived using the T_{n+1} zero-coupon bond as numeraire.

2.1 Valuation of the traffic light option under the forward measure

In this section, the valuation of the traffic light option with T_{n+1} -payoff given by

$$C(S(T_{n+1}), L_n(T_n)) = [\bar{S} - S(T_{n+1})]^+ \cdot [\bar{L} - L_n(T_n)]^+ \quad (4)$$

is considered.³ Choosing the zero-coupon bond maturing at time T_{n+1} as numeraire is very convenient. Inserting in (3) results in

$$\begin{aligned} \pi(t) &= B(t, T_{n+1}) \mathbb{E}^{n+1} \left[\frac{[\bar{S} - S(T_{n+1})]^+ \cdot [\bar{L} - L_n(T_n)]^+}{B(T_{n+1}, T_{n+1})} \mid \mathcal{F}_t \right] \\ &= B(t, T_{n+1}) \\ &\quad \times \mathbb{E}^{n+1} \left[\left[\frac{\bar{S}}{B(T_{n+1}, T_{n+1})} - \frac{S(T_{n+1})}{B(T_{n+1}, T_{n+1})} \right]^+ \cdot [\bar{L} - L_n(T_n)]^+ \mid \mathcal{F}_t \right] \\ &= B(t, T_{n+1}) \mathbb{E}^{n+1} \left[\left[\bar{S} - \frac{S(T_{n+1})}{B(T_{n+1}, T_{n+1})} \right]^+ \cdot [\bar{L} - L_n(T_n)]^+ \mid \mathcal{F}_t \right]. \quad (5) \end{aligned}$$

Since we are working in the lognormal forward model, the LIBOR rate $L_n(t)$ is lognormal under its own measure. The stock index dynamics under the forward measure will depend on the instantaneous development in the zero coupon bond maturing at time T_{n+1} . In our setup this instantaneous development is inconvenient to model, but we know from the FTAP that the discounted stock index process $\frac{S(t)}{B(t, T_{n+1})}$ is a martingale, inducing that it is driftless. It is also noticed that $\frac{S(t)}{B(t, T_{n+1})}$ is actually the forward stock price by the no-arbitrage assumption. Assuming lognormality of the forward stock price process $\frac{S(t)}{B(t, T_{n+1})}$ leaves us with the two stochastic differential equations

$$\begin{aligned} d\left(\frac{S(t)}{B(t, T_{n+1})}\right) &= \left(\frac{S(t)}{B(t, T_{n+1})}\right) \sigma_t dW_{\frac{S}{B}}^{n+1}(t) \\ dL_n(t) &= L_n(t) \lambda_n(t) dW_L^{n+1}(t), \end{aligned} \quad (6)$$

³It may seem strange that the claim is paid at time T_{n+1} since the LIBOR rate is known at time T_n . The reason for this is a technicality coming from the fact that the numeraire, under which the LIBOR rate L_n is a martingale, is the T_{n+1} zero-coupon bond, and we want to use that the time T_{n+1} -value of this is known at any point in time.

This construction is also observed in the market for caplets and floorlets.

where $dW_{\frac{S}{B}}^{n+1}(t) dW_L^{n+1}(t) = \rho_t dt$ and both σ_t and ρ_t are deterministic functions of time. The volatility σ_t of the discounted asset price process can be derived from market prices on ordinary European call options, since a closed form solution given by a Black (1976) formula exists.

$$\begin{aligned}\pi^{call}(t) &= B(t, T) \mathbb{E}^T \left[\frac{(S(T) - K)^+}{B(T, T)} \mid \mathcal{F}_t \right] \\ &= B(t, T) \mathbb{E}^T \left[\left(\frac{S(T)}{B(T, T)} - K \right)^+ \mid \mathcal{F}_t \right] \\ &= S(t) N(d_1) - B(t, T) K N(d_2)\end{aligned}\tag{7}$$

where

$$\begin{aligned}d_1 &= \frac{\ln \frac{S_t}{B(t, T) K} + \frac{1}{2} \sigma_S^2}{\sigma_S} \\ d_2 &= \frac{\ln \frac{S_t}{B(t, T) K} - \frac{1}{2} \sigma_S^2}{\sigma_S}\end{aligned}$$

and

$$\sigma_S^2 = \int_t^T \sigma_u^2 du.$$

By assuming lognormality of the forward stock price process, the volatility of this process can be derived from quoted prices by inversion of (7).

The main result of this section and the first contribution in the paper is given in the Proposition below, where a closed form solution for the price of the traffic light option is stated.

Proposition 1 *Under the assumptions in this section the time t value of the traffic light option with T_{n+1} -payoff (4) is given by*

$$\begin{aligned}
& \pi(S(t), L_n(t), t; \rho_{SL}) \\
&= S(t) L_n(t) \left[\tilde{S} \cdot \tilde{L} \cdot M\left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x}, \frac{\ln \tilde{L} - \mu_y}{\sigma_y}; \rho_{SL}\right) \right. \\
&\quad - \tilde{L} \cdot M\left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \sigma_x, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \rho_{SL}\sigma_x; \rho_{SL}\right) \\
&\quad - \tilde{S} \cdot M\left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \rho_{SL}\sigma_y, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \sigma_y; \rho_{SL}\right) \\
&\quad \left. + e^{\sigma_{xy}} \cdot M\left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \rho_{SL}\sigma_y - \sigma_x, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \rho_{SL}\sigma_x - \sigma_y; \rho_{SL}\right) \right], \quad (8)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{S} &= \frac{\overline{SB}(t, T_{n+1})}{S(t)} & \sigma_x^2 &= \int_t^{T_{n+1}} \sigma_s^2 ds \\
\tilde{L} &= \frac{\overline{L}}{L_n(t)} & \sigma_y^2 &= \int_t^{T_n} \lambda_n(s)^2 ds \\
\mu_x &= -\frac{1}{2} \int_t^{T_{n+1}} \sigma_s^2 ds & \sigma_{xy} &= \int_t^{T_n} \sigma_s \lambda_n(s) \rho_s ds \\
\mu_y &= -\frac{1}{2} \int_t^{T_n} \lambda_n(s)^2 ds & \rho_{SL} &= \frac{\sigma_{xy}}{\sigma_x \sigma_y},
\end{aligned}$$

and $M(\cdot, \cdot; \rho)$ is the standardized bivariate normal distribution function with correlation coefficient ρ .

Proof of the Proposition can be found in Appendix A.

In the traffic light option pricing formula, the bivariate normal distribution has to be evaluated. Several routines exist for this and the one used in this paper can be downloaded from the MatLab central file exchange.⁴

⁴<http://www.mathworks.com/matlabcentral/fileexchange>.

2.2 Valuation of general hybrid products under the spot measure

It is well known that the drift of any asset under the risk neutral measure equals the instantaneous risk free rate r_t , and often dynamics of the form

$$dS_t = r_t S_t dt + \sigma_t S_t dW_S^{\mathbb{Q}}(t) \quad (9)$$

is assumed, with corresponding time t value $S_t = S_0 e^{\int_0^t (r_s - \frac{1}{2}\sigma_s^2) ds + \int_0^t \sigma_s dW_S^{\mathbb{Q}}(s)}$. However, this is not practical since it involves instantaneous rates, and it is more tractable sticking to the LIBOR rates. It is more convenient to find some discrete rate version of the dynamics (9). Instead of using the continuously compounded bank account as the numeraire, the discretely compounded analog introduced by Jamshidian (1997) can be used. The discrete bank account is represented by a rolling strategy in zero coupon bonds. Start with one dollar at time 0 and invest in $\frac{1}{B(0, T_0)}$ zero coupon bonds maturing at time T_0 paying $\frac{1}{B(0, T_0)}$. Invest this amount in bonds which mature at the next tenor date T_1 and continue this strategy until time t . The value of this self financing strategy at any time t is given by

$$\begin{aligned} B^d(t) &= \frac{B(t, T_{i(t)})}{B(0, T_0)} \prod_{j=0}^{i(t)-1} \frac{B(T_j, T_j)}{B(T_j, T_{j+1})} \\ &= \frac{B(t, T_{i(t)})}{B(0, T_0)} \prod_{j=0}^{i(t)-1} (1 + \tau_{j+1} L_j(T_j)), \end{aligned} \quad (10)$$

where $i(t) = \inf \{k \mid T_{k-1} \leq t < T_k\}$. It is the value of the trading strategy at time $T_{i(t)}$, which is then discounted back to time t by multiplying with $B(t, T_{i(t)})$. For $T_0 = 0$ and $t = T_k$ for some k , the discrete bank account reduces to $B^d(T_k) = \prod_{j=0}^{k-1} (1 + \tau_{j+1} L_j(T_j))$.

Assuming that the bond prices have dynamics

$$dB(t, T_j) = B(t, T_j) (a(t, T_j) dt + b(t, T_j) dW(t)) \quad (11)$$

under some underlying probability measure, then using Itô's lemma on $B^d(t)$

gives

$$dB^d(t) = B^d(t) \left(a(t, T_{i(t)}) dt + b(t, T_{i(t)}) dW(t) \right). \quad (12)$$

Again, it is seen that the dynamics involve instantaneous drift and diffusion terms from the bond dynamics. Modeling this can be avoided by letting the claims that have to be priced be tied to the settlement dates of the LIBOR rates, which is not a restrictive assumption.

Determining the discretely compounded analog of the asset price process directly reveals

$$S_t = S_0 \frac{B(t, T_{i(t)})}{B(0, T_0)} \prod_{j=0}^{i(t)-1} (1 + \tau_{j+1} L_j(T_j)) e^{\int_0^t \sigma_s dW_S^d(s) - \frac{1}{2} \int_0^t \sigma_s^2 ds}. \quad (13)$$

Denote the equivalent martingale measure $\mathbb{Q}^d$ corresponding to the discrete bank account as numeraire the spot LIBOR measure. The discounted asset price process $\frac{S(t)}{B^d(t)}$ is then a martingale under $\mathbb{Q}^d$

$$\frac{S(t)}{B^d(t)} = \mathbb{E}^d \left[\frac{S(T)}{B^d(T)} \mid \mathcal{F}_t \right],$$

and from (10) and (13) it is seen that the defined asset price process discounted $\frac{S(t)}{B^d(t)}$ (by construction) has the martingale property.

When pricing under the spot LIBOR measure, the dynamics of the LIBOR rates L_i for $i = 1, \dots, n$ have to be found in order to calculate the expectation

$$\pi(t) = B^d(t) \mathbb{E}^d \left[\frac{1}{B^d(T_i)} C(S(T_0), \dots, S(T_i), L_0(T_0), \dots, L_n(T_i)) \mid \mathcal{F}_t \right]. \quad (14)$$

These dynamics are derived in the lognormal LIBOR market model in Jamshidian (1997) and given by

$$dL_i(t) = L_i(t) \sum_{j=i(t)}^i \frac{\tau_{j+1} L_j(t) \rho_{i,j} \lambda_j(t)}{1 + \tau_{j+1} L_j(t)} \lambda_i(t) dt + L_i(t) \lambda_i(t) dW_i^d(t), \quad (15)$$

where W_i^d is a Wiener process under the spot LIBOR measure $\mathbb{Q}^d$, and $\rho_{i,j}$ is the correlation coefficient between the Wiener processes W_i^d and W_j^d .⁵

⁵Here, the dynamics are stated in the 1-dimensional version. For a version where all the individual LIBOR rates are allowed to be driven by multivariate Brownian components and

Notice how the model is only completely determined on the tenor dates of the LIBOR rates. This can be seen from equation (14), where the time t price depends on the discrete bank account at time t . As also noted in Jamshidian (1997), a simple linear interpolation between the two nearest tenor dates is suggested to get $B^d(t)$ if the time t price of the derivative is needed.

2.3 Instantaneous volatilities and correlation

When evaluating equation (14), the simultaneous distribution of the various stochastic variables under the expectation is required, and this is in general not possible to find analytically, meaning that the evolution in the corresponding processes has to be implemented by simulation. Here, the instantaneous correlations between the different LIBOR rates and the stock index are a fundamental issue, when the simulation is implemented. The correlation matrix stems from

$$\begin{bmatrix} dW_1^d(t) \\ \vdots \\ dW_n^d(t) \\ dW_S^d(t) \end{bmatrix} \begin{bmatrix} dW_1^d(t) & \cdots & dW_n^d(t) & dW_S^d(t) \end{bmatrix} = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} & \rho_{1,S} \\ \rho_{2,1} & 1 & \cdot & \cdot & \rho_{2,S} \\ \vdots & \cdot & 1 & \cdot & \vdots \\ \rho_{n,1} & \cdot & \cdot & 1 & \rho_{n,S} \\ \rho_{S,1} & \cdot & \cdot & \rho_{S,n} & 1 \end{bmatrix} dt. \quad (16)$$

In this implementation, correlations between the LIBOR rates are described by deterministic functions depending on the length between the corresponding tenor dates, T_i, T_j . With correlations involving the stock index, the dependence is on the length between time t and the corresponding tenor date of the LIBOR rate T_i .

For our purpose, it is satisfactory to specify the instantaneous volatility of the LIBOR rates following Rebonato (2002) p. 159 as

$$\lambda_i(t) = f(T_i - t) g(T)$$

and he argues that the functional form of f

$$f(T_i - t) = (a + (T_i - t)b) e^{-(T_i - t)c} + d \quad (17)$$

a complete proof, the reader is referred to Jamshidian (1997). A nice sketch of the proof is given in Glasserman (2003).

is flexible enough to capture desirable criteria such as being hump shaped.⁶

The instantaneous correlation matrix between the LIBOR rates should fulfill four criteria. Namely:

1. Symmetry: $\rho_{i,j} = \rho_{j,i} \forall i, j$.
2. It should be positive semidefinite ($x^T \rho x \geq 0 \forall x \in \mathbb{R}^{\mathbb{N}}$).
3. 1 on the diagonal ($\rho_{i,i} = 1$).
4. Entries in the interval $[-1, 1]$.

Further, we will model it as a time homogeneous function for $T_i, T_j > t$ and $i \neq j$. A simple correlation function that satisfies the requirements above is

$$\rho_{i,j}(t) = e^{-\beta|T_i - T_j|}, \quad (18)$$

where $\beta > 0$.

The only missing aspect now is how to specify the volatility of the stock index, and how it correlates with the LIBOR rates. For simplicity, we will let the volatility be constant ($\sigma_s = \sigma$). It is reasonable to let the Wiener process for the stock index $W_S^d(t)$ be correlated most with the LIBOR rates with shortest distance to maturity $T_i - t$. A convenient form is

$$\rho_{S,i}(t) = \frac{1 - \exp\left(-\frac{\alpha}{(t - T_i - \gamma)}\right)}{1 + \exp\left(-\frac{\alpha}{(t - T_i - \gamma)}\right)} = \tanh\left(\frac{\alpha}{2(t - T_i - \gamma)}\right), \quad (19)$$

where $\gamma > 0$. Positive values of α give rise to negative correlations and vice versa. This specific choice of function ensures correlations between -1 and 1 , and if more flexibility is needed, additional parameters can be included inside the brackets in (19). Of course, for $T_i < t$ the LIBOR rate has matured and the correlation is set to zero.

⁶See Rebonato (2002) p. 167.

3 Numerical Implementation

Valuation of (14) is done in this section, and in the first part the T_{n+1} -claim given by

$$C(S(T_{n+1}), L_n(T_n)) = [\bar{S} - S(T_{n+1})]^+ \cdot [\bar{L} - L_n(T_n)]^+$$

is considered. This resembles the payout in Jørgensen (2007) and is illustrated in Figure 1. The only difference is that the payout is tied to a LIBOR rate instead of some benchmark interest rate depending on the instantaneous short rate process. Notice, also, that since the LIBOR rate sets 6 months prior to expiry, the traffic light option reduces to a plain vanilla equity put option the final 6 months.

The volatility structure of the LIBOR rates is assumed to have the form

$$\begin{aligned}\lambda_i(t) &= f(T_i - t) \\ &= (a + (T_i - t)b) e^{-(T_i - t)c} + d\end{aligned}$$

with parameters taken from Brigo and Mercurio (2006) p. 320

$$a = 0, \quad b = 0.29342753, \quad c = 1.25080230, \quad d = 0.13145869,$$

and it is illustrated in Figure 2.

Let the coefficient in (18) be given by $\beta = 0.1$ as illustrated in Figure 3.

What the correct values are for the α and γ parameters in the function for measuring the correlation between the stock index and the LIBOR rates in equation (19) is unclear. Here α and γ is chosen equal to one giving rise to the correlation curve in Figure 4. Table 1 illustrates instantaneous correlation values as a function of both α , γ and the distance to maturity $T_i - t$ of the corresponding LIBOR rate. The first observation is that the correlation is a decreasing function of distance to maturity for this particular choice of correlation function. This is also a reasonable property. It is the rate maturing nearest from now that reacts most to the market information also driving the stock market. The α parameter controls the level of the correlation and higher absolute values of this parameter increases the absolute correlation across maturities, though not in a parallel way. The γ parameter controls the curvature

of the function. This is also clear from looking at the table, where it is seen that the absolute decrease in correlation as distance to maturity increases is highest for small values of γ . Or loosely stated, the starting point of the function in Figure 4 shifts closer to zero for higher γ values.

3.1 Pricing of the traffic light option with Proposition 1

In Figure 5, the price of the traffic light option is found with Proposition 1, and depicted as a function of the stock price and the LIBOR rate at time zero. In Figure 6, the influence of the correlation on the price is illustrated. All parameters are set as mentioned under the Figures and the instantaneous LIBOR rate volatility in the calculations is specified as in Figure 2.

From Figure 5, it is seen how the traffic light option value is a decreasing function of both the stock index price and the initial value of the LIBOR rate, which was also expected from the specification of the payout function.

Further, it is seen that the option value converges towards zero, when the underlying variables move further out-of-the money, and the value is highest for the underlying variables deep in-the money. Figure 6 also shows (not surprisingly) how the price is increasing in correlation. With higher correlation, there is a higher probability of realizing simultaneous drops in both the stock index and the LIBOR rate which reflects in a higher option price. It is clearly confirmed from the Figure, how the correlation choice is a very important issue when the traffic light option is priced. The difference between the highest and the lowest price is as much as 260% of the zero-correlation price.

In Table 2, the dependence of the price of an "at the money" traffic light option is shown with respect to the instantaneous correlation and time to maturity of the option.

For example, when time to maturity is 3 years, then it means that the underlying LIBOR rate matures after $2\frac{1}{2}$ years. From the Table, the same relationship as depicted above is seen: The correlation has a positive effect on the price of the option. With respect to time to maturity, it is seen that the price reaches its maximum somewhere between 5 and 15 years for various correlation levels, and as time to maturity converges towards infinity, the price converges towards zero. This hump-shaped behavior as a function of time to maturity is also shared with individual floorlet end put option prices in this

model, so the fact that it is seen with respect to the traffic light option is not surprising.

3.2 Pricing with simulation

In practice it is possible to price any European type T_i -payoff given by $C(S(T_0), \dots, S(T_i), L_0(T_0), \dots, L_n(T_i))$ by simulation. In this section a multi-factor model is implemented.⁷ Illustration of the model is performed via the pricing of a hybrid derivative issued by BNP Paribas and known as the EUR Sage Note with a maturity of $T_n = 15$ years and with a yearly coupon payment given by

$$C(S_{T_i}, L_{T_i+2}(T_i), L_{T_i+10}(T_i)) = \min\{\max\{k_i(L_{T_i+10} - L_{T_i+2}), 0.01\}, 0.08\}, \quad (20)$$

where the gearing k_i is high if the EuroStoxx 50 price index performs well, benefitting investors who believe that the equity market will rise.⁸ The gearing is given by the relation

$$k_i = \begin{cases} 2 & S_{T_i} < 0.95S_0 \\ 4 & 0.95S_0 \leq S_{T_i} < 2S_0 \\ 8 & S_{T_i} \geq 2S_0 \end{cases}.$$

The coupon is the spread between a long and a short LIBOR rate geared by k_i , floored by 1% and capped by 8%. The rationale underlying an investment in a product like this is that the gearing is higher than what can be achieved with a normal steepener for the same price if the equity market rises substantially. Payments similar to this form is being increasingly used in the construction of structured products. The pricing is performed under the spot LIBOR measure.

In a simulation of the LIBOR rates, the first choice is to fix a time grid of future time points over which to simulate $0 = t_0 < t_1 < \dots < t_m < t_{m+1}$. In this time grid, it is convenient to let the tenor dates $T_0 < T_1 < \dots < T_n$ be a subset. Further, by letting the time difference between two simulation points be constant ($t_{j+1} - t_j = \delta$), the notation is reduced.

⁷A total of $n + 1$ factors drive the model here.

⁸This derivative is described in BNP Paribas (2006), although with a slightly different coupon. There it is the spread between two CMS rates that are geared by k_i . Other examples of hybrid products can be found in Hunter and Picot (2005).

Simulating the LIBOR rates with an Euler scheme on $\log \widehat{L}_i$ results in⁹

$$\widehat{L}_i(t_{j+1}) = \widehat{L}_i(t_j) \cdot \exp \left\{ \left(\mu_i(t_j) - \frac{1}{2} \lambda_i(t_j)^2 \right) \delta + \sqrt{\delta} \lambda_i(t_j) Z_{j+1} \right\}, \quad (21)$$

with

$$\mu_i(t_j) = \sum_{l=i(t_j)}^i \frac{\tau_{l+1} \widehat{L}_l(t_j) \rho_{i,l} \lambda_l(t_j)}{1 + \tau_{l+1} \widehat{L}_l(t_j)} \lambda_i(t_j),$$

and $Z_1, \dots, Z_{m+1}$ independent $N(0, 1)$ random variables. In the equations above, the hats have been added to clarify that the continuous LIBOR rates have been discretized. The simulation is initialized with (1) by setting

$$\widehat{L}_i(0) = \frac{1}{\tau_{i+1}} \left(\frac{B(0, T_i)}{B(0, T_{i+1})} - 1 \right), \quad i = 1, \dots, n.$$

The value of the bank account at maturity T_n can be derived from the simulation of the LIBOR rates using (10). The final variable to simulate is the stock price given by $S(T_n) = S_0 B^d(T_n) e^{\int_0^{T_n} \sigma_s dW_s^d(s) - \frac{1}{2} \int_0^{T_n} \sigma_s^2 ds}$, where $X(T_n) = S_0 \cdot e^{\int_0^{T_n} \sigma_s dW_s^d(s) - \frac{1}{2} \int_0^{T_n} \sigma_s^2 ds}$ can be simulated with the scheme

$$\widehat{X}(t_{j+1}) = \widehat{X}(t_j) \cdot \exp \left\{ -\frac{1}{2} \sigma_s^2 \delta + \sqrt{\delta} \sigma_s Z_{j+1} \right\}.$$

And finally, correlation is achieved by simulating at every time step from an $(n+1)$ -dimensional normal distribution with correlation matrix (16), and then use these normal variables in the discretizations above.

By the use of the discretizations above, the time zero price of the derivative

$$\pi(0) = \mathbb{E}^d \left[\sum_{i=1}^{15} \frac{1}{B^d(T_i)} C(S_{T_i}, L_{T_i+2}(T_i), L_{T_i+10}(T_i)) \right], \quad (22)$$

can then be approximated with Monte Carlo simulation.

In Figure 7, a plot of the hybrid derivative price is graphed as a function of the parameters α and γ controlling the correlations. In the simulation, the time intervals between the tenor dates are set constant equal to one half year ($\tau_i = 0.5$) and the time step used in the Euler discretization is a quarter of a year ($\delta = 0.25$). The starting term structure is assumed flat and all the

⁹See Glasserman (2003) for more details.

LIBOR rates $L_i(0)$ is assumed to have the same initial value. Finally, each point in the graph is calculated with 200,000 replications. The value of the derivative is highest for high values of α and low values of γ when $\alpha > 0$. This is explained with Table 1, where it is seen that low values of γ induce that the stock index correlates (in absolute value) more with the LIBOR rate in the short end. For positive values of α , this results in negative correlations. *Ceteris paribus*, this in turn induces a higher probability of observing high returns on the stock market in combination with a greater decrease in rates in the short end than in the long end. All together, this results in higher expected coupon payments on the EUR Sage Note giving rise to a higher price. For $\alpha < 0$, the effect of γ is opposite, since a low value of γ here entails that positive movements in the stock market influence positively short rates more than long rates. This makes the realization of negative spreads in combination with high stock market returns more likely, which again produces lower expected coupon payments. It is also observed that α is the most significant variable. From Table 1, it is also clear that α is the most influential correlation parameter.

Not all values of α and γ are allowed in the simulation, since the correlation matrix (16) has to be positive semidefinite. This is the reason why the α and γ axes in Figure 7 do not cover the same intervals as in Table 1.

4 Conclusion

In the paper, it was shown how the pricing of a new type of derivatives known as hybrid derivatives with payoffs depending on both a stock index and some LIBOR rates can be implemented in a LIBOR market model framework. The main advantage of using a LIBOR market framework is that the pricing is based on observable market quantities which makes the model easy to calibrate.

In some cases, a closed derivative pricing formula exists and the first contribution of this paper was the derivation of an analytically closed solution for the price of the traffic light option, when the underlying discounted stock price process and LIBOR rate are assumed lognormal under the forward measure. It was shown how the price of the derivative is a decreasing function of the initial value of the underlying variables, and it was illustrated how the correlation between the stock index and the LIBOR rate is of major importance. This is also the key obstacle when using Proposition 1, where all the parameters

can be extracted from observed market data except for the correlation. The correlation between stocks and LIBOR rates is thus a natural research question to delve further into.

It was shown how the pricing of general hybrid products can be performed under the spot LIBOR measure. The dynamics of the stock index under this measure were derived, which to the knowledge of the author, is new. In many cases, pricing has to be done by Monte Carlo simulation, and the implementation of the model was illustrated via the pricing of a hybrid product known as the EUR Sage Note. Payments of this form are used extensively in the construction of structured products, and the approach described in this paper can be used for the pricing of these payoffs.

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A Proof of Proposition 1

Proof. Under the forward measure $\mathbb{Q}^{n+1}$ conditioned on $\mathcal{F}_t$ both $\frac{S(T_{n+1})}{B(T_{n+1}, T_{n+1})}$ and $L_n(T_n)$ are lognormally distributed meaning that $\frac{S(T_{n+1})}{B(T_{n+1}, T_{n+1})} = \frac{S(t)}{B(t, T_{n+1})} e^X$ and $L_n(T_n) = L_n(t) e^Y$, where (X, Y) is bivariate normally distributed, independent of $\mathcal{F}_t$ and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \right)$$

with $\mu_x = -\frac{1}{2} \int_t^{T_{n+1}} \sigma_s^2 ds$, $\mu_y = -\frac{1}{2} \int_t^{T_n} \lambda_n(s)^2 ds$, $\sigma_x^2 = \int_t^{T_{n+1}} \sigma_s^2 ds$, $\sigma_y^2 = \int_t^{T_n} \lambda_n(s)^2 ds$ and $\sigma_{xy} = \int_t^{T_n} \sigma_s \lambda_n(s) \rho_s ds$.

Rewriting the expectation for the payoff (5) results in;

$$\begin{aligned} & B(t, T_{n+1}) \mathbb{E}^{n+1} \left[\left[\bar{S} - \frac{S(t)}{B(t, T_{n+1})} e^X \right]^+ \cdot [\bar{L} - L_n(t) e^Y]^+ \mid \mathcal{F}_t \right] \\ &= S(t) L_n(t) \\ &\times \left\{ \tilde{S} \tilde{L} \mathbb{E}^{n+1} \left[1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \mid \mathcal{F}_t \right] - \tilde{L} \mathbb{E}^{n+1} \left[e^X 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \mid \mathcal{F}_t \right] \right. \\ &\quad \left. - \tilde{S} \mathbb{E}^{n+1} \left[e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \mid \mathcal{F}_t \right] + \mathbb{E}^{n+1} \left[e^X e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \mid \mathcal{F}_t \right] \right\} \\ &= S(t) L_n(t) \left\{ \tilde{S} \tilde{L} \mathbb{E}^{n+1} \left[1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] - \tilde{L} \mathbb{E}^{n+1} \left[e^X 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] \right. \\ &\quad \left. - \tilde{S} \mathbb{E}^{n+1} \left[e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] + \mathbb{E}^{n+1} \left[e^X e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] \right\}. \quad (23) \end{aligned}$$

The first equality holds because $\tilde{S}$ and $\tilde{L}$ are both measurable with respect to $\mathcal{F}_t$ and the second because of the independence between the variables and the σ -algebra.

Finding the price of the traffic light option then reduces to evaluation of the four expectations in (23). Starting from the beginning:

1. If f denotes the density function for the standard bivariate normal dis-

tribution, then the first expectation reduces to

$$\begin{aligned}
\mathbb{E}^{n+1} \left[1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] &= \int_{-\infty}^{\ln \tilde{S}} \int_{-\infty}^{\ln \tilde{L}} f(x, y) dy dx \\
&= \int_{-\infty}^{\ln \tilde{S}} \int_{-\infty}^{\ln \tilde{L}} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp \left\{ -\frac{1}{2(1-\rho^2)} \right. \\
&\quad \times \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x} \right) \left(\frac{y-\mu_y}{\sigma_y} \right) + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] \Big\} dy dx \\
&= M \left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x}, \frac{\ln \tilde{L} - \mu_y}{\sigma_y}; \rho_{SL} \right),
\end{aligned}$$

where the fourth equality follows by substituting $u = \frac{x-\mu_x}{\sigma_x}$ and $v = \frac{y-\mu_y}{\sigma_y}$.

2. For the second we have that

$$\mathbb{E}^{n+1} \left[e^X 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] = \int_{-\infty}^{\ln \tilde{L}} \int_{-\infty}^{\ln \tilde{S}} e^x f(y) f(x | y) dx dy.$$

Rewriting the exponent gives

$$\begin{aligned}
&x - \frac{1}{2} \left(\frac{y-\mu_y}{\sigma_y} \right)^2 - \frac{1}{2\sigma_x^2(1-\rho^2)} \left(x - \mu_x - \frac{\rho\sigma_x}{\sigma_y} (y - \mu_y) \right)^2 \\
&= \mu_x + \frac{1}{2}\sigma_x^2 - \frac{1}{2(1-\rho^2)} (u^2 - 2\rho uv + v^2),
\end{aligned}$$

where the substitutions $u = \frac{x-\mu_x}{\sigma_x} - \sigma_x$ and $v = \frac{y-\mu_y}{\sigma_y} - \rho\sigma_x$ have been made. This again gives

$$\begin{aligned}
&\mathbb{E}^{n+1} \left[e^X 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] \\
&= e^{\mu_x + \frac{1}{2}\sigma_x^2} \cdot M \left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \sigma_x, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \rho_{SL}\sigma_x; \rho_{SL} \right).
\end{aligned}$$

3. The third expectation can be calculated in the same way as the second.

$$\begin{aligned}
&\mathbb{E}^{n+1} \left[e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] \\
&= e^{\mu_y + \frac{1}{2}\sigma_y^2} \cdot M \left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \rho_{SL}\sigma_y, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \sigma_y; \rho_{SL} \right).
\end{aligned}$$

4. Finally,

$$\mathbb{E}^{n+1} \left[e^X e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] = \int_{-\infty}^{\ln \tilde{S}} \int_{-\infty}^{\ln \tilde{L}} e^x e^y f(x) f(y | x) dy dx.$$

Rewriting the exponent again gives

$$\begin{aligned} & x + y - \frac{1}{2} \left(\frac{x - \mu_x}{\sigma_x} \right)^2 - \frac{1}{2\sigma_y^2(1-\rho^2)} \left(y - \mu_y - \frac{\rho\sigma_y}{\sigma_x} (x - \mu_x) \right)^2 \\ &= \mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_y^2 + \rho\sigma_x\sigma_y - \frac{1}{2(1-\rho^2)} (u^2 - 2\rho uv + v^2), \end{aligned}$$

where the substitutions $u = \frac{x - \mu_x}{\sigma_x} - \rho\sigma_y - \sigma_x$ and $v = \frac{y - \mu_y}{\sigma_y} - \rho\sigma_x - \sigma_y$ have been made. Together the above finally results in

$$\begin{aligned} & \mathbb{E}^{n+1} \left[e^X e^Y 1_{\{e^X < \tilde{S}\}} 1_{\{e^Y < \tilde{L}\}} \right] \\ &= e^{\mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_y^2 + \rho\sigma_x\sigma_y} \times \\ & M \left(\frac{\ln \tilde{S} - \mu_x}{\sigma_x} - \rho_{SL}\sigma_y - \sigma_x, \frac{\ln \tilde{L} - \mu_y}{\sigma_y} - \rho_{SL}\sigma_x - \sigma_y; \rho_{SL} \right). \end{aligned}$$

Now collecting all the terms, and observing that for our particular claim $\mu_x = -\frac{1}{2}\sigma_x^2$ and $\mu_y = -\frac{1}{2}\sigma_y^2$, the desired result is achieved. ■

B Tables

Table 1: The instantaneous correlation between the stock index and the LIBOR rates for various values of γ , α and for three different values of distance to maturity of the rates.

	$\alpha \backslash \gamma$	0.0005	0.25	0.5	1	2	3
$T_i - t = 0$	-5.0	1.00	1.00	1.00	0.99	0.85	0.68
	-2.0	1.00	1.00	0.96	0.76	0.46	0.32
	-0.5	1.00	0.76	0.46	0.24	0.12	0.08
	0.0	0.00	0.00	0.00	0.00	0.00	0.00
	0.5	-1.00	-0.76	-0.46	-0.24	-0.12	-0.08
	2.0	-1.00	-1.00	-0.96	-0.76	-0.46	-0.32
	5.0	-1.00	-1.00	-1.00	-0.99	-0.85	-0.68
$T_i - t = 2.5$	-5.0	0.76	0.72	0.68	0.61	0.50	0.42
	-2.0	0.38	0.35	0.32	0.28	0.22	0.18
	-0.5	0.10	0.09	0.08	0.07	0.06	0.05
	0.0	0.00	0.00	0.00	0.00	0.00	0.00
	0.5	-0.10	-0.09	-0.08	-0.07	-0.06	-0.05
	2.0	-0.38	-0.35	-0.32	-0.28	-0.22	-0.18
	5.0	-0.76	-0.72	-0.68	-0.61	-0.50	-0.42
$T_i - t = 7.5$	-5.0	0.32	0.31	0.30	0.29	0.26	0.23
	-2.0	0.13	0.13	0.12	0.12	0.10	0.10
	-0.5	0.03	0.03	0.03	0.03	0.03	0.02
	0.0	0.00	0.00	0.00	0.00	0.00	0.00
	0.5	-0.03	-0.03	-0.03	-0.03	-0.03	-0.02
	2.0	-0.13	-0.13	-0.12	-0.12	-0.10	-0.10
	5.0	-0.32	-0.31	-0.30	-0.29	-0.26	-0.23

Table 2: Traffic light option prices $\cdot 100$ for various parameter values of the instantaneous correlation and time to maturity. The parameters are set to: $S_0 = \bar{S} = 100, L_n(0) = \bar{L} = 0.04, \tau = 0.5, \sigma_s = 0.2$ and the LIBOR rate volatility as in Figure 2. The initial term structure is assumed flat.

$\rho \backslash T_{n+1}$	1	2	3	5	10	15	20	30	100
-1.00	0.201	0.166	0.129	0.095	0.054	0.030	0.016	0.005	0.000
-0.75	0.402	0.657	0.786	0.930	1.051	1.031	0.953	0.747	0.057
-0.50	0.643	1.304	1.687	2.113	2.515	2.536	2.385	1.905	0.149
-0.25	0.922	2.069	2.760	3.518	4.243	4.299	4.051	3.237	0.252
0.00	1.234	2.939	3.978	5.108	6.182	6.261	5.891	4.689	0.355
0.25	1.578	3.908	5.335	6.875	8.319	8.408	7.890	6.243	0.456
0.50	1.956	4.984	6.842	8.834	10.675	10.759	10.064	7.911	0.552
0.75	2.371	6.181	8.524	11.022	13.294	13.353	12.439	9.687	0.632
1.00	2.827	7.534	10.444	13.526	16.256	16.219	14.990	11.480	0.683

C Figures

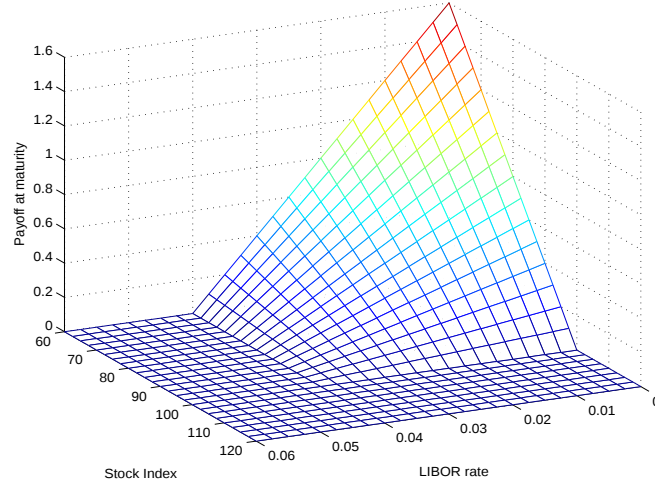


Figure 1: The payout of the traffic light option with $\bar{S} = 100$ and $\bar{L} = 0.04$.

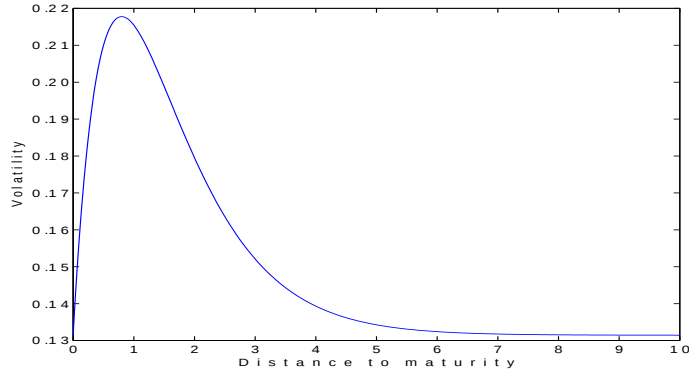


Figure 2: The instantaneous LIBOR rate volatility as a function of $T_i - t$.

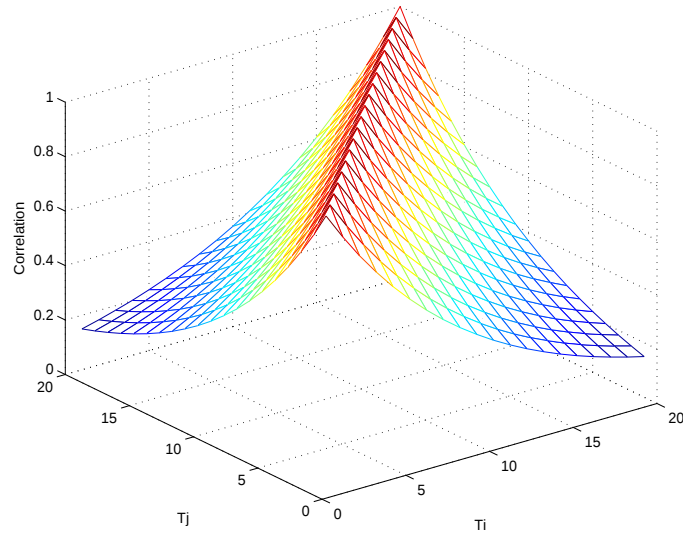


Figure 3: The correlations between the LIBOR rates illustrated for $\beta = 0.1$.

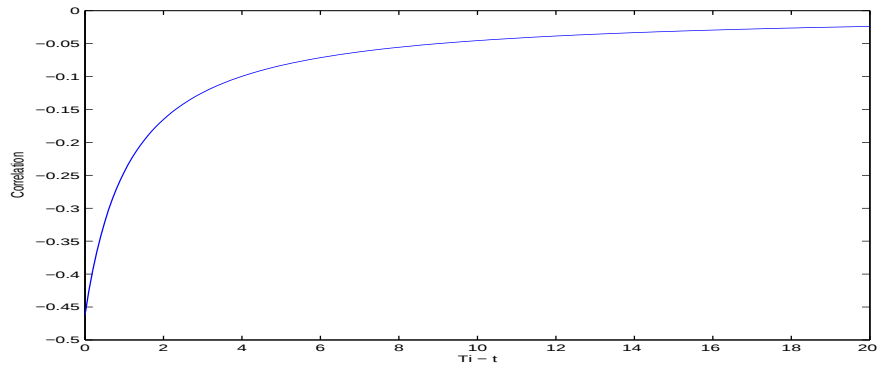


Figure 4: Here the correlation between the stock index and the LIBOR rates as in equation (19) is depicted with $\gamma = \alpha = 1$.

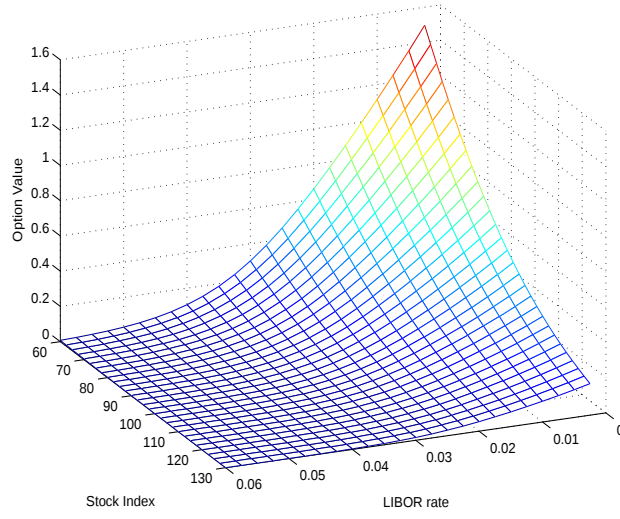


Figure 5: The traffic light option price as a function of the stock index and the LIBOR rate. The parameter values are $\bar{S} = 100$, $\bar{L} = 0.04$, $T_{n+1} = 3$, $\rho = -0.5$, $\sigma_s = \sigma = 0.2$ and the term structure is assumed flat equal to the initial LIBOR rate.

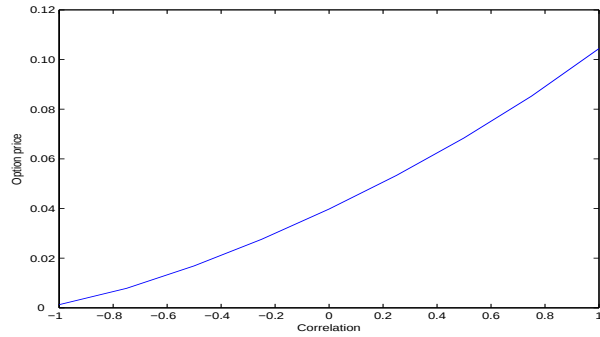


Figure 6: The traffic light option price as a function of the correlation with $S_0 = 100$, $L_n(0) = 0.04$, $\bar{S} = 100$, $\bar{L} = 0.04$, $T_{n+1} = 3$, $\sigma_s = \sigma = 0.2$ and $B(0, T_{n+1}) = 0.8890$.

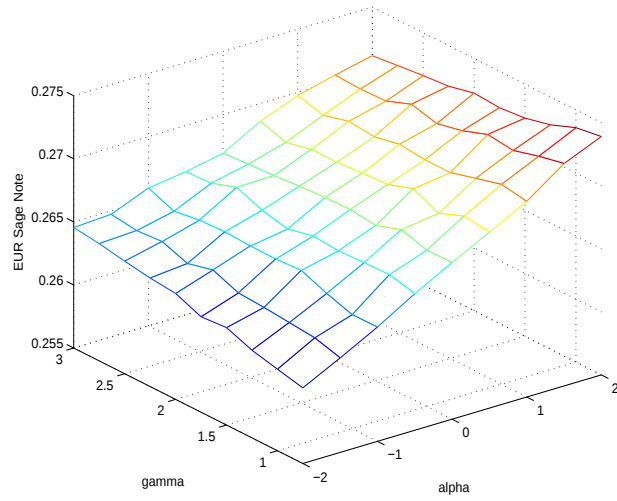


Figure 7: The value of the hybrid derivative as a function of α and γ . The parameters are: $S_0 = 100$, $\tau = \frac{1}{2}$, $\delta = \frac{1}{4}$, $\sigma_s = 0.2$, $\beta = 0.1$, the LIBOR rate volatilities as in Figure 2 and the initial term structure is assumed flat with all the LIBOR rates equal to 0.04.

Sato Processes in Default Modeling

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Abstract

In reduced form default models, the instantaneous default intensity is the classical modeling object. Survival probabilities are then given by the Laplace transform of the cumulative hazard defined as the integrated intensity process. Instead, recent literature tends to specify the cumulative hazard process directly.

Within this framework we present a new model class where cumulative hazards are described by self-similar additive processes, also known as Sato processes. Furthermore, we analyze specifications obtained via a simple deterministic time-change of a homogeneous Lévy process. While the processes in these two classes share the same average behavior over time, the associated intensities exhibit very different properties.

Concrete specifications are calibrated to data on all the single names included in the iTraxx Europe index. The performances are compared with those of the classical CIR intensity and a recently proposed class of intensity models based on Ornstein-Uhlenbeck type processes. It is shown that the time-inhomogeneous Lévy models achieve comparable calibration errors with fewer parameters, and with more stable parameter estimates over time. However, the calibration performance of the Sato processes and the time-change specifications are practically indistinguishable.

Key words: Credit default swap, reduced form model, Sato process, time-changed Lévy process, cumulative hazard.

1 Introduction

In recent years the market for credit derivatives has experienced a remarkable development. The estimated total notional value of the credit default swap market in the US rose from \$900 billion in 2000 to the breathtaking number of more than \$60 trillion in the beginning of 2008 - roughly twice the size of the entire US stock market. As a consequence of the financial crisis, the market moved in the opposite direction through 2008 and 2009 and was down to a total notional value of \$30 trillion by March 2009. This development has fueled the interest among finance scholars, and the academic literature concerning the modeling of corporate default risk has experienced tremendous growth. For a comprehensive overview of credit risk models see e.g. Lando (2004), and Bielecki and Rutkowski (2002). The reduced form models developed by Jarrow and Turnbull (1995), Jarrow et al. (1997), Lando (1998), Duffie and Singleton (1997, 1999), Madan and Unal (1998), and Elliott et al. (2000) have been particularly popular among practitioners. In this framework the default time of a firm is modeled as the first jump time of an underlying process, typically described by a Cox process with stochastic intensity rate λ . The survival probabilities up to time t are then given as

$$\mathbb{Q}(\tau > t) = \mathbb{E} [e^{-A_t}] \quad (1)$$

with the process A defined as the integrated instantaneous intensity

$$A_t = \int_0^t \lambda_s ds \quad (2)$$

and with intensity rate λ often described by a positive and stationary affine process. This intensity-based approach has become quite popular and widely used in practice with the classical Cox, Ingersoll, and Ross (1985) (CIR) model as a standard benchmark. Extensions allowing both for jumps and shifts have been proposed in Brigo and Alfonsi (2005), Brigo and Cousot (2006), and Brigo and El-Bachir (2009) for the purpose of pricing credit default swap (CDS) options. Furthermore, hedging of CDS options in the CIR model is discussed in Bielecki et al. (2008a) and in a more general setting in Bielecki et al. (2008b).

However, problems are still present when considering parameter stability over time, leading to large variations in sensitivities and hedge parameters,

which is problematic from a risk management perspective.

The intensity-based approach is generalized from a theoretical point of view in Jeanblanc and Rutkowski (2000, 2002), where the modeling object is the process A in (1) as opposite to the instantaneous intensity rate λ . In particular A , which we will refer to as the *cumulative hazard process*, is no longer constrained to be absolutely continuous w.r.t. the Lebesgue measure.

Recently, cumulative hazard processes displaying jumps have been adopted in the context of multivariate modeling by e.g. Joshi and Stacey (2006), Di Graziano and Rogers (2006), and Hull and White (2008). However, their performances on single-name derivatives have not been empirically investigated, focus being on matching market-inferred correlations.

Partly motivated by the ongoing financial turmoil which has almost entirely wiped out the market for multivariate credit instruments, we focus on credit default swap pricing. We present two new model classes where the cumulative hazard process A belongs to:

1. The class of self-similar additive processes introduced by Sato (1991) and therefore termed Sato processes. Sato processes have been employed for option pricing and interest rate modeling by e.g. Carr et al. (2005, 2007), Eberlein and Madan (2009), and Skovmand (2008).
2. The class of time-changed Lévy processes obtained via the evaluation of a homogeneous Lévy process at a re-scaled point in time t^γ . While sharing with Sato processes the same average time behavior, cumulative hazards in this model class are not self-similar and exhibit rather different characteristics.

Both families of processes are extremely convenient for a number of reasons. First, they display enormous flexibility in terms of distribution modeling. They are analytically tractable and they allow for closed form expressions for default probabilities hereby enabling straightforward calibration to CDS spreads. Finally, they allow for more flexibility and non-linearity in the long term behavior of the cumulative hazard in contrast to the more traditional framework described in (2), where the expression below holds if the intensity λ is ergodic

$$t^{-1} \int_0^t \lambda_s ds \rightarrow \bar{\lambda} \quad \text{as } t \rightarrow \infty$$

with $\bar{\lambda}$ denoting the long-run average of λ .

The alternative concrete specifications here selected for calibration to CDS market data are obtained by setting the unit-time law of the cumulative hazard as either a Gamma or an Inverse Gaussian distribution. Both these distributions are well known in the financial literature and have been widely used in the fields of econometrics and derivative pricing by Madan and Seneta (1990), Rydberg (1999), Barndorff-Nielsen and Shephard (2001), and Nicolato and Venardos (2003) among others.

Cariboni and Schoutens (2009) employ both Gamma and Inverse Gaussian Ornstein-Uhlenbeck (OU) type processes to describe instantaneous default intensities. They find that the OU type intensity models and the CIR intensity model display analogous performances when pricing CDSs. These models are equally parsimonious in the number of parameters and are therefore included as comparative benchmarks for the cumulative hazard models proposed here.

In order to assess both pricing capabilities and parameter stability over time, all the models are calibrated to daily observations of the single names included in the iTraxx Europe Series 8 index in the period from September 17, 2007 to March 14, 2008.

The rest of the paper is structured as follows: Section 2 describes the general cumulative hazard modeling framework. Sato specifications are analyzed in Section 3 while time-changed Lévy processes are considered in Section 4. Section 5 illustrates the benchmark models and the calibration results are presented in Section 6. Last, Section 7 concludes.

2 The General Cumulative Hazard Modeling Framework

We start by describing in some detail the modeling approach adopted here, which is known as the Cox construction of a default time. A comprehensive and unifying theoretical analysis of general reduced form models is given in Bielecki and Rutkowski (2002), and Jeanblanc and Le Cam (2007).

Consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{F}_t), \mathbb{Q})$ which is large enough to support a càdlàg, adapted process $A = (A_t)$ and a mean-one exponential random variable E_1 , which is independent of $\mathcal{F}_\infty$. Furthermore, the process

A , termed the cumulative hazard (CH), has (strictly) increasing paths, and $A_0 = 0$. The default time τ is then defined as the first time A reaches or is above the level of an independent exponential random variable

$$\tau = \inf \{t > 0 \mid A_t \geq E_1\}, \quad (3)$$

and the survival probabilities are given by (1). As already mentioned, the classical intensity-based models introduced in Lando (1998) are obtained by specifying the CH process as the integrated intensity $A_t = \int_0^t \lambda_s ds$. However, in this general framework A is no longer constrained to be absolutely continuous w.r.t. the Lebesgue measure.

The directing filtration $(\mathcal{F}_t)$ can be seen as carrying the default-free information while the knowledge about whether default has occurred is contained in the enlarged filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,$$

where $\mathcal{H}_t = \sigma \{ \mathbb{1}_{\{\tau \leq s\}} : 0 \leq s \leq t \}$ is the filtration generated by the default process ($H_t = \mathbb{1}_{\{\tau \leq t\}}$). Within this framework the so-called immersion property holds, i.e. any $(\mathcal{F}_t)$ -local martingale is a $(\mathcal{G}_t)$ -local martingale (see Brémaud and Yor (1978)).

If the compensator $\Lambda^G = (\Lambda_t^G)$ (w.r.t the filtration $(\mathcal{G}_t)$) of the process H is absolutely continuous w.r.t the Lebesgue measure

$$\Lambda_t^G = \int_0^t \lambda_s^G ds,$$

the default time is said to have *intensity rate*, or simply *intensity*, λ^G and it holds that

$$\lambda_t^G = \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{Q}(t < \tau \leq t + h \mid \mathcal{G}_t).$$

The compensator Λ^G can be related to the CH process A as follows

$$\Lambda_t^G = \int_0^{t \wedge \tau} \frac{dC_s}{e^{-A_{s-}}},$$

where the process $C = (C_t)$ is the $(\mathcal{F}_t)$ -compensator of the process $1 - e^{-A}$. Hence, if C is absolutely continuous w.r.t. the Lebesgue measure, $C_t = \int_0^t c_s ds$,

the default time has intensity

$$\lambda_t^G = \mathbb{1}_{\{t < \tau\}} \lambda_t \quad \text{with} \quad \lambda_t = \frac{c_t}{e^{-A_t-}} \quad (4)$$

and with a slight abuse of terminology, the $(\mathcal{F}_t)$ -adapted process λ is also referred to as the intensity process. Notice that if the CH A is specified as in the classical intensity-based approach then the rate of growth λ is indeed the intensity process in the sense of equation (4).

For the actual computations, it is convenient to express the intensity process λ in terms of the semimartingale characteristics of the CH process A (see Jacod and Shiryaev (1987), Chapter 2 for the definition of semimartingale characteristics).

Proposition 1 *Assume that the cumulative hazard process has absolutely continuous characteristics, i.e. relatively to the zero truncation function it has the following canonical representation*

$$A_t = \int_0^t d_s ds + x * \mu, \quad (5)$$

where the random measure $\mu(\omega, dx, dt)$ associated to the jumps of A has absolutely continuous compensator

$$\nu(\omega, dx, dt) = K_t(\omega, dx) dt.$$

Then the default time defined in (3) admits intensity λ , which is given by

$$\lambda_t = d_t - \int_0^{+\infty} (e^{-x} - 1) K_t(\omega, dx). \quad (6)$$

Furthermore, if A is a process with independent increments, i.e. an additive process, the intensity λ is deterministic and given by

$$\lambda_t = -\frac{d}{dt} \log \mathbb{E}[e^{-A_t}]. \quad (7)$$

Proof. First, we compute the compensator C of the process $1 - e^{-A}$. By change of variable formula, we obtain that

$$1 - e^{-A_t} = e^{-A_t-} \cdot L_t$$

with

$$L_t = A_t - (e^{-x} - 1 + x) * \mu.$$

Applying the canonical representation (5) we can rewrite L as follows

$$\begin{aligned} L_t &= x * \mu + \int_0^t d_s ds - (e^{-x} - 1 + x) * \mu \\ &= -(e^{-x} - 1) * (\mu - \nu) + \int_0^t d_s ds - (e^{-x} - 1) * \nu, \end{aligned}$$

from which it follows that

$$C_t = \int_0^t e^{-A_{s-}} \left(d_s - \int_0^{+\infty} (e^{-x} - 1) K_s(\omega, dx) \right) ds$$

and the result (6) now follows from expression (4).

If furthermore A is additive, the drift process d_t and the measures $K_t(dx)$ (also known as the *Lévy system*) are deterministic. Then, expression (7) follows immediately by recalling the Lévy-Kintchine representation for the Laplace transform

$$\mathbb{E} [e^{uA_t}] = \exp \int_0^t \left(u d_s + \int_0^\infty (e^{ux} - 1) K_s(dx) \right) ds. \quad (8)$$

■

We conclude this section by recalling the so-called *hazard based pricing rule* (see Elliott et al. (2000)): given a claim $X \in \mathcal{F}_T$, it holds that

$$\mathbb{E}[X \mathbf{1}_{\{T < \tau\}} | \mathcal{G}_t] = \mathbf{1}_{\{\tau > t\}} e^{A_t} \mathbb{E}[e^{-A_T} X | \mathcal{F}_t].$$

In particular, assuming no recovery payment and that the instantaneous interest rate process $r = (r_t)$ is adapted to the default-free filtration $(\mathcal{F}_t)$, we obtain that the price $B(t, T)$ of a corporate bond is given by

$$B(t, T) = \mathbf{1}_{\{\tau > t\}} e^{A_t} \mathbb{E} \left[e^{-(A_T + \int_t^T r_s ds)} | \mathcal{F}_t \right].$$

By assuming independence between A and the interest rate process r , the

expression further reduces to

$$\begin{aligned} B(t, T) &= p(t, T) \mathbf{1}_{\{\tau > t\}} e^{A_t} \mathbb{E} [e^{-A_T} | \mathcal{F}_t] \\ &= p(t, T) \mathbb{Q}(\tau > T | \mathcal{G}_t), \end{aligned}$$

where $p(t, T)$ is the price of a risk-free zero coupon bond with maturity T .

3 Sato Specifications

We start by recalling that a stochastic process $A = (A_t)$ is called self-similar if for any $\alpha > 0$ there exists a $\beta > 0$ such that for all t

$$A_{\alpha t} \stackrel{\text{law}}{=} \beta A_t, \tag{9}$$

which implies that a change of time scale has an analogous effect as a change in the spatial scale. From definition (9) it follows that there exists a $\gamma > 0$, the *exponent of self-similarity*, such that $\beta = \alpha^\gamma$. A well-known class of self-similar processes is given by the strictly stable processes, i.e. processes with independent and homogeneous increments, with unit-time law described by a strictly stable distribution. Sato (1991) introduced the broader class of self-similar additive processes, here termed *Sato processes*, as processes satisfying (9) and with independent but not necessarily homogeneous increments. Sato processes are intimately related to the class of self-decomposable distributions. More precisely, the unit-time law of a Sato process is necessarily self-decomposable. Vice versa, for any given self-decomposable law X and an exponent $\gamma > 0$ there exists a self-similar additive process with unit-time law given by X and exponent of self-similarity given by γ . (For more information on Sato processes see e.g. Sato (1999)).

When X is set as a self-decomposable law with support given by the positive real line, we obtain that the corresponding Sato process has increasing paths. Therefore it is suitable to describe a CH process, here termed *Sato CH*.

Self-decomposability requires that the cumulant generating function of the unit-time law X

$$\kappa_X(u) = \log \mathbb{E}[e^{uX}]$$

takes the following representation

$$\kappa_X(u) = du + \int_0^\infty (e^{ux} - 1) \frac{h(x)}{x} dx, \quad (10)$$

where $d \geq 0$, and $h(x)$ is a nonnegative decreasing function on $(0, +\infty)$ satisfying

$$\int_0^{+\infty} (1 \wedge x) \frac{h(x)}{x} dx < +\infty. \quad (11)$$

Self-similarity implies that

$$\mathbb{E} [e^{uA_t}] = \mathbb{E} [e^{ut^\gamma X}] = e^{\kappa_X(ut^\gamma)}. \quad (12)$$

Carr et al. (2007) computed the precise form of the Lévy-Kintchine representation of A_t , which is given as in (8) with drift specified as

$$d_t = d \gamma t^{\gamma-1} \quad (13)$$

and Lévy system

$$K_t(dx) = -\frac{h'(\frac{x}{t^\gamma})}{t^{\gamma+1}} \gamma dx, \quad x > 0. \quad (14)$$

Inspection of the Lévy system $K_t(dx)$ reveals that the behavior of the function $h(x)$ in (10) determines whether the corresponding Sato CH process A displays finite or infinite activity. More precisely, Carr et al. (2005) showed that A jumps finitely or infinitely often on finite time intervals if $h(0) < +\infty$ or $h(0) = +\infty$, respectively.

By Proposition 1, a Sato CH process allows for the existence of a deterministic intensity rate λ . In virtue of (12), the exact expression becomes

$$\lambda_t = \gamma t^{\gamma-1} \kappa'_X(-t^\gamma). \quad (15)$$

Moreover, if the unit-time law X has finite first moment, the corresponding Sato CH process A has average behavior given by

$$\mathbb{E}[A_t] = \mathbb{E}[X] t^\gamma. \quad (16)$$

This allows for an increasing, constant or decreasing average growth rate corresponding to $\gamma > 1$, $\gamma = 1$ or $\gamma < 1$, respectively. However, the effect of the

exponent of self-similarity γ on the long term behavior of the intensity λ is not completely analogous. In the case of a driftless Sato CH process with finite activity, the intensity vanishes over time for any choice of γ . A more complete description is given in the following Proposition.

Proposition 2 *Let A be a Sato CH process with exponent of self-similarity γ and unit-time law X specified as in (10), and let λ denote the associated intensity. Then the following holds:*

- i) *If $0 < \gamma < 1$ then $\lambda_t \rightarrow 0$ as $t \rightarrow +\infty$.*
- ii) *If $\gamma \geq 1$, $d = 0$ and $h(0) < +\infty$ then $\lambda_t \rightarrow 0$ as $t \rightarrow +\infty$.*
- iii) *Finally if $d > 0$ then*

$$\lambda_t \rightarrow \begin{cases} \text{constant} \geq d & \text{if } \gamma = 1 \\ +\infty & \text{if } \gamma > 1 \end{cases}$$

as $t \rightarrow +\infty$.

Proof. The intensity can be expressed as follows

$$\lambda_t = \gamma t^{\gamma-1} \left(d + \int_0^{+\infty} e^{-t^\gamma x} h(x) dx \right) \quad (17)$$

by combining (10) and (15). Then all the statements above follow immediately from the integrability property (11) of the function $h(x)$. ■

When dealing with a driftless Sato CH process of infinite activity and exponent of self-similarity $\gamma \geq 1$, the long term behavior of the intensity depends on the particular shape of the function $h(x)$, and it is not possible to provide a general statement (see e.g. the case of an IG-Sato process discussed below).

From (12) it follows that the unconditional survival probabilities are given by the following simple expression

$$\mathbb{Q}(\tau > t) = \mathbb{E} \left[e^{-A_t} \right] = e^{\kappa_X(-t^\gamma)}. \quad (18)$$

Therefore, if the cumulant generating function κ_X is known explicitly, the survival probabilities can be computed in closed-form enabling straightforward calibration of the model to credit default swaps. This is the case for the two

concrete specifications here considered, namely the *Sato-Gamma* and the *Sato-IG* cumulative hazards.

The *Sato-Gamma CH* is obtained by specifying the unit-time distribution X as a Gamma law $\Gamma(a, b)$ with density function given by

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \quad x > 0. \quad (19)$$

The cumulant generating function is given by

$$\kappa_X(u) = \log \left(\frac{1}{(1 - u b^{-1})^a} \right) = \int_0^\infty (e^{ux} - 1) \frac{a e^{-bx}}{x} dx \quad (20)$$

and in view of (10) we see that a Gamma law is self-decomposable. Moreover, the associated Sato-Gamma CH displays finite activity and its intensity is given by

$$\lambda_t = a\gamma \frac{t^{\gamma-1}}{b + t^\gamma}.$$

The *Sato-IG CH* is obtained by setting the unit-time distribution X as an Inverse Gaussian law $IG(a, b)$ with the following density function

$$f_X(x) = \frac{a}{\sqrt{2\pi}} e^{ab} x^{-\frac{3}{2}} e^{-(b^2 x + \frac{a^2}{x})/2} \quad x > 0. \quad (21)$$

The $IG(a, b)$ distribution is also self-decomposable as can be seen from the cumulant generating function which has the following expression¹

$$\kappa_X(u) = ab - a\sqrt{b^2 - 2u} = \int_0^\infty (e^{ux} - 1) \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}b^2 x}}{x^{\frac{3}{2}}} dx. \quad (22)$$

We see that a Sato-IG CH is a process of infinite activity and the corresponding intensity is given by

$$\lambda_t = a\gamma \frac{t^{\gamma-1}}{\sqrt{b^2 + 2t^\gamma}}.$$

Notice that as $t \rightarrow +\infty$, the intensity converges to 0, a positive constant or $+\infty$ if the exponent of self-similarity γ is chosen smaller than, equal to or greater than 2, respectively.

¹The self-decomposability of Inverse Gaussian distributions was first proved by Halgreen (1979).

4 Time-Changed Lévy Specifications

A deterministic time-change of a Lévy process enable us to construct CHs which share with Sato processes the same average time behavior (16), but otherwise display different characteristics. Let $L = (L_t)$ be an increasing Lévy process, i.e. a process with positive, time-homogeneous and independent increments. By changing the time scale of L in the following manner

$$A_t = L_{t^\gamma} \quad \gamma > 0, \quad (23)$$

we obtain an increasing additive CH process, which we will refer to as a *TC Lévy* CH process. Once again, let X denote the positive unit-time law of A and $\kappa_X(u)$ its cumulant generating function. Only infinite divisibility of X and not necessarily self-decomposability is required in the TC Lévy construction (23). Therefore, $\kappa_X(u)$ takes the general form

$$\kappa_X(u) = du + \int_0^\infty (e^{ux} - 1)\xi(dx), \quad (24)$$

where $d \geq 0$ and the Lévy measure $\xi(dx)$ satisfies $\int_0^{+\infty} (1 \wedge x)\xi(dx) < +\infty$. The TC Lévy process A clearly displays the same activity level of the subordinand process L , i.e. it is a process of finite activity if L is a compound Poisson process while it exhibits infinite activity if $\int_0^{+\infty} \xi(dx) = +\infty$. It also follows immediately that

$$\mathbb{E} [e^{uA_t}] = e^{t^\gamma \kappa_X(u)},$$

and that the Lévy-Kintchine representation is given as in (8), with the drift specified as

$$d_t = d \gamma t^{\gamma-1}$$

and the Lévy system given by

$$K_t(dx) = \gamma t^{\gamma-1} \xi(dx).$$

Hence, by Proposition 1 the default time associated with a TC Lévy CH admits intensity λ , which is given by the power function

$$\lambda_t = -\kappa_X(-1) \gamma t^{\gamma-1}. \quad (25)$$

Notice that unlike the case of Sato specifications, the average growth rate $d\mathbb{E}[A_t]/dt$ (when defined) and the intensity λ of a TC Lévy CH display the same behavior over time, independently of the choice of the unit-time law X , albeit for a scaling constant.

The survival probabilities are given by

$$\mathbb{Q}(\tau > t) = \mathbb{E}[e^{-A_t}] = e^{t^\gamma \kappa_X(-1)} \quad (26)$$

and can be computed in closed form whenever κ_X is known. In consistence with the concrete models in Section 3, we calibrate to market data the *TC Lévy-Gamma* and the *TC Lévy-IG* processes obtained by selecting the $\Gamma(a, b)$ law (19) and the $IG(a, b)$ law (21) for the unit-time distribution X . Expressions (20) and (22) show that both the TC Lévy-Gamma and TC Lévy-IG processes display infinite activity.

5 Intensity Rate Models

In this section we briefly illustrate the intensity rate models which will be used as benchmarks to assess the performances of the CH models described above.

5.1 Ornstein-Uhlenbeck Type Intensity

Cariboni and Schoutens (2009) have recently proposed a new default model in which the intensity rate is governed by a positive process of the Ornstein-Uhlenbeck type (OU type process henceforth). A positive OU type process $\{\lambda_t : t > 0\}$ is defined as the solution of the stochastic differential equation

$$d\lambda_t = -\theta\lambda_t dt + dZ_{\theta t} \quad \lambda_0 > 0, \quad (27)$$

where θ is a positive real number and $Z = (Z_t)$ is a purely jumping, increasing Lévy process often termed the *Background Driving Lévy Process* (BDLP henceforth). The process λ is mean-reverting, it increases only by jumps and between jumps it decays exponentially. Under (mild) integrability conditions on the BDLP Z , the OU type process λ described in (27) is stationary with invariant distribution, which does not depend on the mean reversion parameter θ due to the unusual timing of $Z_{\theta t}$. See Sato (1999) for further details and a

thorough analysis of OU type processes.

As in Cariboni and Schoutens (2009) we examine the two concrete stationary OU type processes, denoted by $\text{Gamma}(a, b)\text{-OU}$ and $\text{IG}(a, b)\text{-OU}$, characterized by having invariant law given by the $\Gamma(a, b)$ and the $\text{IG}(a, b)$ distributions respectively.

The $\text{Gamma}(a, b)\text{-OU}$ specification is obtained by choosing the BDLP Z as a compound Poisson process with Lévy density

$$w(x) = a b \exp(-bx) ,$$

i.e. Z is a compound Poisson process with intensity a and jump-size distributed as an exponential law with parameter b .

The $\text{IG}(a, b)\text{-OU}$ process is obtained by specifying the Lévy density of Z as

$$w(x) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}} (1 + b^2 x) e^{-\frac{1}{2} b^2 x} .$$

In this case Z is not a compound Poisson process, and therefore the intensity λ jumps infinitely often in finite time intervals. See Barndorff-Nielsen and Shephard (2001) for a complete derivation of the results above.

In both the $\text{Gamma}(a, b)\text{-OU}$ and the $\text{IG}(a, b)\text{-OU}$ case, the Laplace transform of the CH $A_t = \int_0^t \lambda_s ds$ is available in terms of elementary functions providing explicit expressions for the survival probabilities. Having specified λ as a $\text{Gamma}(a, b)\text{-OU}$ process, one obtains

$$\mathbb{E}[e^{u A_t}] = \exp \left(\frac{u \lambda_0}{\theta} (1 - e^{-\theta t}) + \frac{\theta a}{u - \theta b} \left(b \log \left(\frac{b}{b - u \theta^{-1} (1 - e^{-\theta t})} \right) - ut \right) \right) \quad (28)$$

while for the $\text{IG}(a, b)\text{-OU}$ specification, the Laplace transform takes the form

$$\mathbb{E}[e^{u A_t}] = \left(\frac{\lambda_0 u}{\theta} (1 - e^{-\theta t}) + \frac{2au}{b\theta} B(u, t) \right) , \quad (29)$$

where

$$B(u, t) = \frac{1 - \sqrt{1 + v(1 - e^{-\theta t})}}{v} \quad (30)$$

$$+ \frac{1}{\sqrt{1 + v}} \left(\operatorname{arctanh} \left(\frac{\sqrt{1 + v(1 - e^{-\theta t})}}{\sqrt{1 + v}} \right) - \operatorname{arctanh} \left(\frac{1}{\sqrt{1 + v}} \right) \right)$$

$$v = \frac{-2u}{b^2\theta}.$$

For the derivation, see Nicolato and Venardos (2003).

5.2 CIR Intensity

As the final comparison model we choose the CIR intensity, where λ follows a mean reverting process

$$d\lambda_t = \theta (\bar{\lambda} - \lambda_t) dt + \sigma \sqrt{\lambda_t} dW_t, \quad (31)$$

with W representing a standard Brownian motion, θ the rate of mean reversion, $\bar{\lambda}$ the long run level of the intensity, and σ the volatility. Furthermore, the Feller condition is imposed in order to ensure positive intensities

$$2\theta\bar{\lambda} > \sigma^2.$$

Survival probabilities are also known in closed form and given by (see Cox et al. (1985))

$$\mathbb{Q}(\tau > t) = \mathbb{E} \left[e^{-\int_0^t \lambda_s ds} \right] = A(0, t) e^{-B(0, t)\lambda_0}, \quad (32)$$

where

$$A(0, t) = \left(\frac{2he^{(\theta+h)\frac{t}{2}}}{2h + (\theta + h)(e^{th} - 1)} \right)^{\frac{2\theta\bar{\lambda}}{\sigma^2}} \quad (33)$$

$$B(0, t) = \frac{2(e^{th} - 1)}{2h + (\theta + h)(e^{th} - 1)} \quad (34)$$

$$h = \sqrt{\theta^2 + 2\sigma^2}.$$

6 Calibration of the Models

A total of seven CH models have been introduced and discussed: Sato-Gamma, Sato-IG, TC Lévy-Gamma, TC Lévy-IG, Gamma-OU, IG-OU and CIR.

The models are calibrated by matching the market-inferred CDS spreads on each name to the model-inferred spreads.

Recall that a CDS insures the protection buyer against default of an underlying company in exchange for a stream of periodic payments to the protection seller. In case of default, the contract is terminated prematurely, and at the time of default the protection seller pays to the protection buyer the face value of the corporate bond minus a possible recovery R , while the protection buyer pays the premium accrued between the last payment date and the default time.

We will make the simplifying assumption that on average defaults take place midway between two payment dates. Furthermore, assuming no counter party risk and independence between the CH and interest rates, the value of a CDS with maturity T is given by the difference between the discounted protection payment and the discounted quarterly paid CDS spread c

$$V^{CDS} = (1 - R) \int_0^T p(0, s) d\mathbb{Q}(s) - \frac{c}{4} \sum_{i=1}^n \left\{ p(0, T_i) \mathbb{Q}(\tau > T_i) + \frac{1}{2} p\left(0, \frac{T_{i-1} + T_i}{2}\right) \mathbb{Q}(T_{i-1} < \tau \leq T_i) \right\},$$

where $\mathbb{Q}(s) = \mathbb{Q}(\tau \leq s)$ is the default probability up to time s , R is the recovery on the bond, $\{T_i, i = 1, \dots, n\}$ the quarterly payment dates and $\{p(0, T_i), i = 1, \dots, n\}$ the observed default-free zero-coupon bond prices. Consequently, the par spread equals

$$c^* = \frac{(1 - R) \int_0^T p(0, s) d\mathbb{Q}(s)}{\frac{1}{4} \sum_{i=1}^n \left\{ p(0, T_i) \mathbb{Q}(\tau > T_i) + \frac{1}{2} p\left(0, \frac{T_{i-1} + T_i}{2}\right) \mathbb{Q}(T_{i-1} < \tau \leq T_i) \right\}}. \quad (35)$$

Since survival probabilities are given by analytically closed expressions in all the models described, the parameters can be directly calibrated by matching the model spreads in (35) to those observed in the market.

We calibrate by minimizing the sum of squared errors (SE) given by

$$SE = \sum_{CDSs} (c_{Market} - c_{Model})^2$$

using the Nelder-Mead simplex algorithm. Furthermore, we compute the resulting average absolute error as a percentage of the mean price (APE) and the average relative percentage error (ARPE)

$$APE = \frac{1}{\text{mean CDS spread}} \sum_{CDSs} \frac{|c_{Market} - c_{Model}|}{\# \{CDS \text{ prices}\}} \quad (36)$$

$$ARPE = \frac{1}{\# \{CDS \text{ prices}\}} \sum_{CDSs} \frac{|c_{Market} - c_{Model}|}{c_{Market}}. \quad (37)$$

We set the recovery to 0.4 and the interest rate r is chosen equal to 4 percent. Although the assumption of a flat 0.4 recovery is no longer an agreed standard for multi-name derivatives, it is not too restrictive for our analysis as we focus on CDS contracts.

The data consists of daily observations of the 125 single names in the iTraxx Europe Series 8 index, from September 17th, 2007 to March 14th, 2008 (a total of 130 days) in order to check both the pricing capabilities of the models and the stability of the parameters over time.² The models are fitted to spreads for maturities 1,3,5,7 and 10 years. For each observation, the calibration is initialized with the parameters estimated from the previous observation.

For the CIR and the two integrated OU-intensity models, the calibration for all the companies and for a given point in time takes less than a minute. The procedure is even faster for the Sato and TC Lévy specifications, approximately 30 seconds on a 2 GB ram, 2 GHz dual core pc using MatLab.

Figure 1 illustrates the average spreads for the 125 companies in the iTraxx during the period analyzed. The increasing behavior over time clearly depicts the credit crisis which erupted in August 2007 and gained strength through 2008.

We divide our empirical study into two parts; a pricing analysis and a stability analysis.

²September 17th was the first Monday after the roll over of the index and March 14th the last Friday before the next roll over.

6.1 Pricing Analysis

Table 1 reports the spread term structures obtained for three arbitrary companies in the alternative models along with the corresponding ARPE and APE performances after calibration to data from a single day.

Except for the CIR model in the Deutsche Post AG observation, all the models perform satisfactorily with ARPE and APE errors below 3%. The two OU models produce slightly smaller errors. This is not surprising since the OU models comprise 4 parameters as opposed to the 3 parameters involved in the Sato and TC Lévy models. It should also be noticed that the two TC Lévy models achieve spreads almost identical to those of the Sato-IG model.

Table 2 reports the average of the ARPE and APE across the 125 firms and 130 days as well as the maximum calibration errors within the entire data set. As regards the average errors, the Sato-Gamma and the two OU intensity models perform best, the Sato-IG and the two time changed Lévy models follow, and finally the CIR model performs worst. More diversified findings are obtained when studying the maximum errors. The Sato-Gamma model is the best performer when it comes to the maximum ARPE, while the maximum APE seems to slightly favor the OU intensity models. Notice how the CIR model fails severely.

The slightly better fit produced by the Sato-Gamma and the OU-models is confirmed by Figures 2 and 3 where all the 16,250 ($130 \cdot 125$) errors, APE and ARPE, are collected in histograms for the different models. The overall quality of fit is satisfactory, with the Sato-Gamma and the two OU models producing error-distributions shifted a little more to the left in comparison to the Sato-IG and the time changed Lévy models. Finally, we see how the CIR model fails to fit the market on many days.

6.2 Stability Analysis

Besides achieving a good fit to market prices it is desirable that the resulting calibrated parameters do not fluctuate wildly over time, since this will lead to large variations in sensitivities and hedge parameters.

In order to perform the stability analysis, one of the parameters in each of the models has been fixed as otherwise the numerical routine seems to encounter some identification issues. In the Sato and TC Lévy specifications, we set a

to 1, while in the OU-case a is set to 100. Finally, we set $\sigma = 0.1$ in the CIR model. Without this parameter fixation, the pricing errors are reduced slightly, but at the cost of a severe reduction in parameter stability. The values reported for the fixed parameters are based on the average of the results of the unrestricted calibrations performed over time and firms.

The average and maximum errors APE and ARPE obtained under daily re-calibration are reported in Table 3. We observe how the performances of the models are slightly reduced with the CIR model suffering most under the parameter fixation. The distributions of the ARPEs and APEs are omitted since they look almost identical to those reported in Figures 2 and 3, albeit being shifted slightly to the right accordingly to the increased errors.

To examine the stability of the parameters we consider the lag- i autocorrelation for each of the parameter time series as suggested by Cariboni and Schoutens (2009). Recall that the lag- i autocorrelation for a time series $X = \{X_t\}_{t=1,\dots,N}$ is given by

$$\rho_i = \frac{\mathbb{E}[(X_t - \mathbb{E}[X_t])(X_{t+i} - \mathbb{E}[X_t])]}{\sqrt{\mathbb{E}[(X_t - \mathbb{E}[X_t])^2] \mathbb{E}[(X_{t+i} - \mathbb{E}[X_t])^2]}},$$

yielding a suitable measure of stability. For a given parameter in a given model, the lag-1 autocorrelation coefficients obtained for the 125 names in the index are collected in histograms in Figure 4. Notice that the Sato and the TC Lévy models display autocorrelation distributions shifted to the right in comparison to the CIR and the two OU models. The parameters are thus more stable in time. Finally, Figure 5 shows the average autocorrelation for each model parameter across firms for the first five lags. Again, as far as stability is concerned, the Sato and the TC Lévy models outperform the CIR and the two OU models. However, when comparing Sato with TC Lévy models it is not possible to determine the better model class. The four models are equally stable in their optimal parameters. In fact, when sharing the same unit-time distribution, the corresponding Sato and TC Lévy models produce almost indistinguishable calibrated parameters, despite the different characteristics of the CH processes employed.

7 Conclusion

In this paper we have proposed and analyzed two new reduced form models where the cumulative hazard belongs to the class of Sato and deterministic TC Lévy processes, respectively. Both model classes are tractable, as they allow for closed-form survival probabilities. They are very parsimonious as only 2-3 parameters are employed and display cumulative hazards with the same flexible and non-linear average long run behavior. However, the two model classes exhibit very different properties, e.g. in the activity level and in the behavior of the associated intensities. Moreover, TC Lévy processes do not possess the self-similarity property which characterizes Sato specifications.

From each class two concrete specifications have been empirically investigated and compared to two OU type intensity models and the CIR model. All the models are calibrated to daily observations of credit default swap spreads on the names constituting the iTraxx Europe series 8 index. Except for the CIR intensity case, the models produce comparable and satisfactory calibration errors, perhaps slightly favoring the Sato-Gamma. Moreover, the Sato and the TC Lévy models require one parameter less and display more stable parameter estimates over time. Still, based on our stability results, it is not possible to discern whether Sato models are preferable to TC Lévy models. Thus, our analysis indicates that the flexible average behavior shared by the two model classes is the critical factor in their satisfactory empirical performances.

However, it should be mentioned that in both the Sato and the TC Lévy models, forward CDS spreads are deterministic functions of time since the cumulative hazard is a process with independent increments. This is clearly a shortcoming as CDS spread volatilities can be rather large in the market (see e.g. Hull and White (2003)). Further research could move towards accommodating this limitation, e.g. by modeling the CH as a stochastically time-changed Sato process, hereby introducing state dependence while retaining the average long term behavior.

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A Tables

Table 1: The spreads from the models calibrated on Thursday, January 3rd, 2008 along with the corresponding errors ARPE and APE.

Company	Model	1YR	3YR	5YR	7YR	10YR	ARPE(%)	APE(%)
Banco Espirito Santo SA	Market	37.90	45.00	51.50	54.20	55.00		
	Sato-Gamma	37.43	46.49	50.74	53.31	55.62	1.75	1.73
	Sato-IG	37.95	46.05	50.24	53.13	56.25	1.83	1.92
	TC Lévy-Gamma	37.95	46.04	50.24	53.13	56.25	1.83	1.92
	TC Lévy-IG	37.95	46.05	50.24	53.13	56.25	1.83	1.92
	Gamma-OU	37.65	46.08	50.71	53.42	55.75	1.48	1.50
	IG-OU	37.65	46.08	50.71	53.42	55.75	1.48	1.50
	CIR	37.65	46.08	50.71	53.42	55.75	1.48	1.50
Endesa SA	Market	44.40	63.70	74.50	81.30	88.30		
	Sato-Gamma	44.37	63.81	74.36	81.36	88.30	0.10	0.10
	Sato-IG	45.28	62.23	73.53	80.96	89.30	1.12	1.04
	TC Lévy-Gamma	45.28	62.23	73.52	80.96	89.31	1.12	1.04
	TC Lévy-IG	45.28	63.23	73.52	80.96	89.30	1.12	1.04
	Gamma-OU	44.41	63.68	74.45	81.42	88.24	0.06	0.07
	IG-OU	44.50	63.35	74.63	81.69	88.02	0.35	0.35
	CIR	44.50	63.35	74.63	81.69	88.02	0.35	0.35
Deutsche Post AG	Market	12.50	24.80	37.20	45.30	55.00		
	Sato-Gamma	11.86	25.94	36.58	45.14	55.14	2.39	1.54
	Sato-IG	12.51	25.93	36.13	44.71	55.64	2.00	1.97
	TC Lévy-Gamma	12.51	25.93	36.13	44.71	55.64	2.00	1.97
	TC Lévy-IG	12.51	25.93	36.13	44.71	55.64	2.00	1.97
	Gamma-OU	12.20	25.77	36.48	45.10	55.25	1.84	1.40
	IG-OU	12.20	25.71	36.52	45.20	55.18	1.69	1.24
	CIR	14.17	24.72	34.74	44.16	57.04	5.31	4.23

Table 2: Average and maximum errors across all companies and dates.

Model	Sato-Gamma	Sato-IG	TC Lévy-Gamma	TC Lévy-IG	Gamma-OU	IG-OU	CIR
Mean ARPE	2.95	4.23	5.17	5.17	2.80	2.78	5.75
Mean APE	2.19	3.33	3.98	3.98	2.23	2.22	4.34
Max ARPE	16.52	22.98	23.19	23.19	24.63	24.63	93.00
Max APE	14.70	14.18	15.72	15.72	13.97	13.97	41.77

Table 3: Average and maximum errors across all companies and dates with one parameter fixed in each model.

Model	Sato-Gamma	Sato-IG	TC Lévy-Gamma	TC Lévy-IG	Gamma-OU	IG-OU	CIR
Mean ARPE	4.56	5.17	5.17	5.17	2.83	3.01	10.38
Mean APE	3.47	3.98	3.98	3.98	2.26	2.44	7.43
Max ARPE	21.92	23.19	23.19	23.19	24.64	24.63	93.01
Max APE	15.49	15.71	15.72	15.72	13.97	13.97	92.78

B Figures

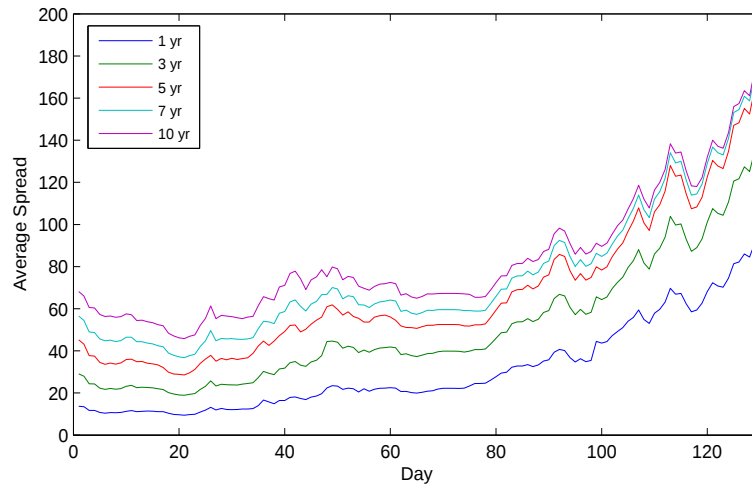


Figure 1: Depicted time series of the average of the 125 CDS spreads of the names in the iTraxx for the various maturities covering the period from September 17th, 2007 to March 14th, 2008.

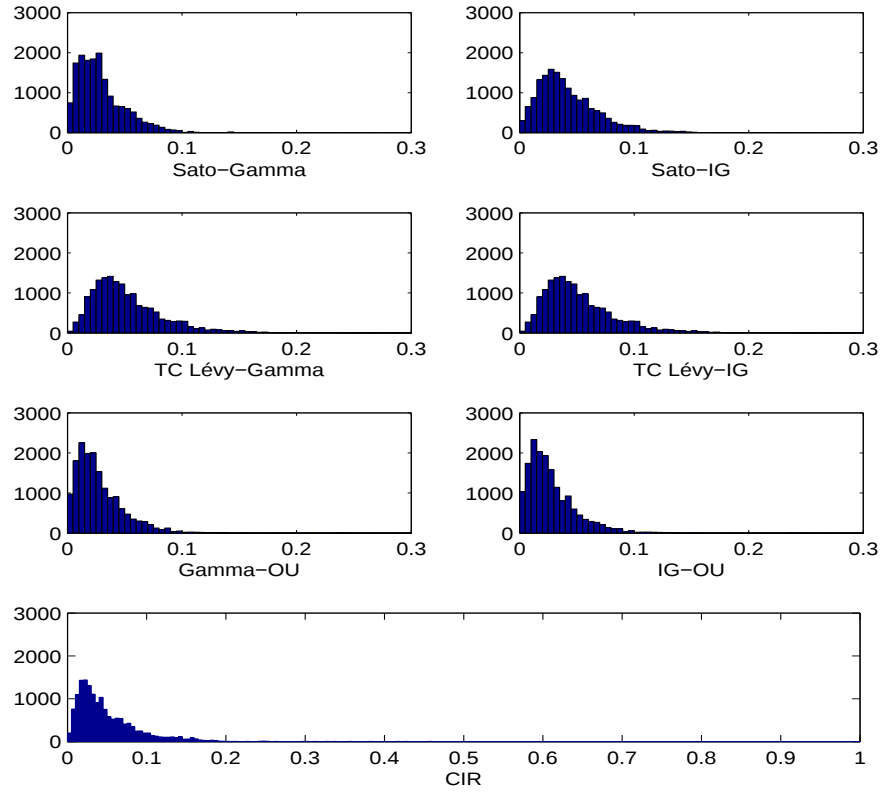


Figure 2: Distributions of ARPE across companies and dates for the different models.

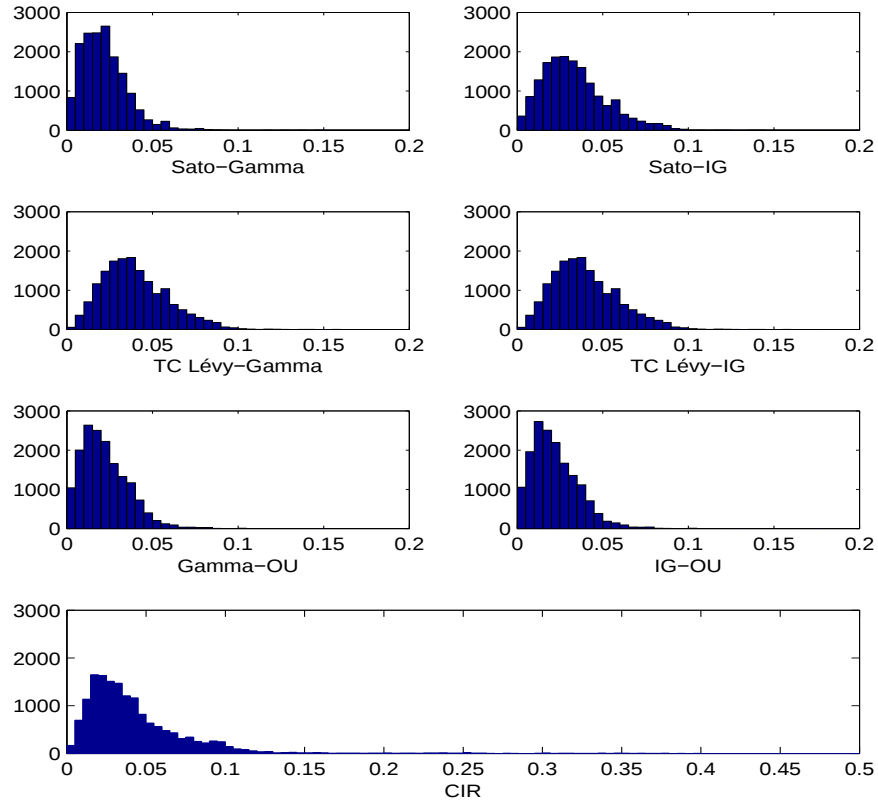


Figure 3: Distributions of APE across companies and dates for the different models.

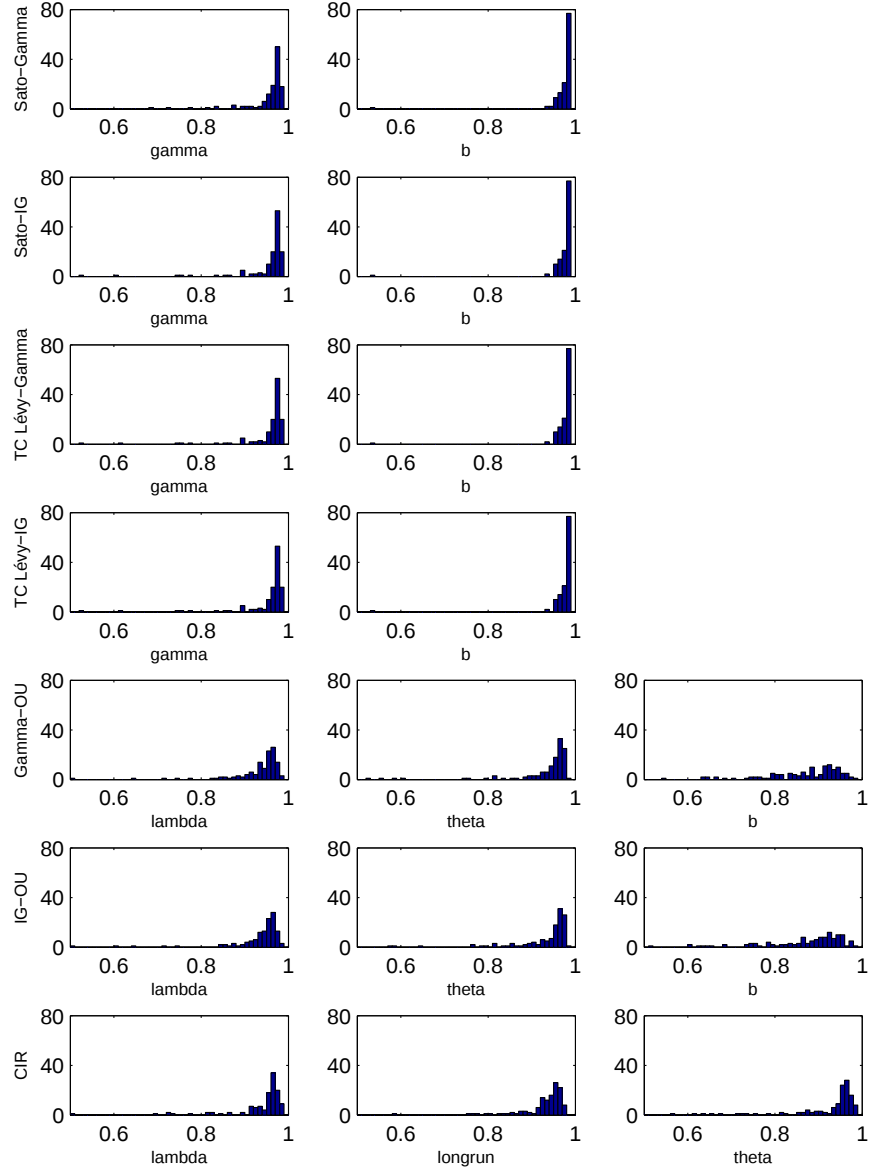


Figure 4: The 1-lag autocorrelation distributions for the parameters in the different models. Each row corresponds to a specific model.

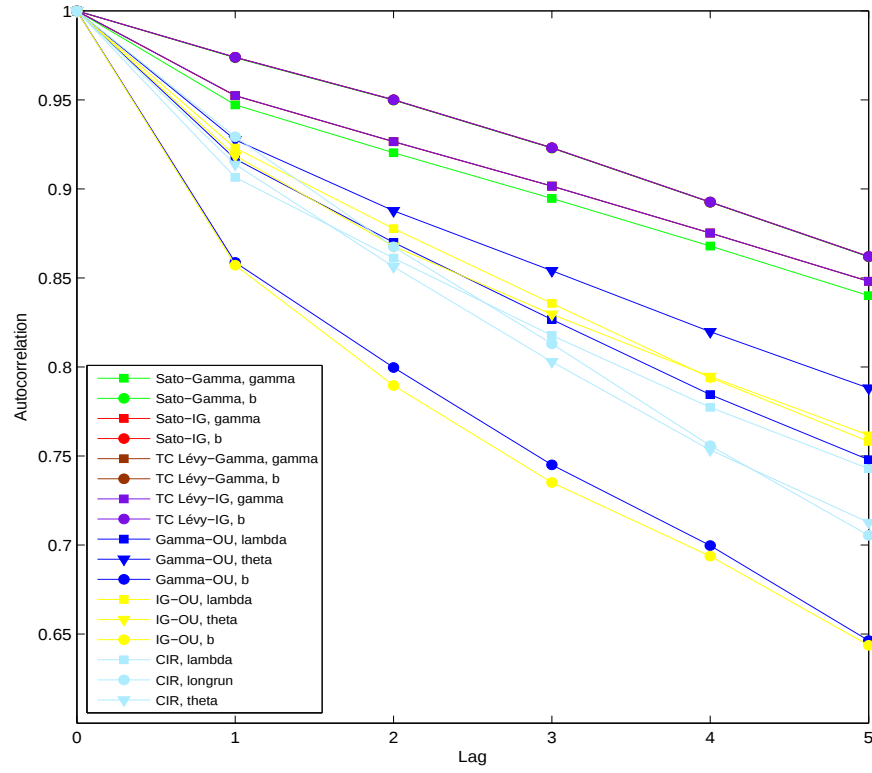


Figure 5: The average of the autocorrelation across firms for each parameter in each model as a function of the lag.

A Consistent Pricing Model for Index Options and Volatility Derivatives *

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Abstract

We propose and study a flexible modeling framework for the joint dynamics of an index and a set of forward variance swap rates written on this index, allowing volatility derivatives and options on the underlying index to be priced consistently. Our model reproduces various empirically observed properties of variance swap dynamics and allows for jumps in volatility and returns.

An affine specification using Lévy processes as building blocks leads to analytically tractable pricing formulas for options on the VIX as well as efficient numerical methods for pricing of European options on the underlying asset. The model has the convenient feature of decoupling the vanilla skews from spot/volatility correlations and allowing for different conditional correlations in large and small spot/volatility moves.

We show that our model can simultaneously fit prices of European options on S&P 500 across strikes and maturities as well as options on the VIX volatility index. The calibration of the model is done in two steps, first by matching VIX option prices and then by matching prices of options on the underlying.

Key words: Variance swap, volatility derivatives, Lévy process, VIX index, option pricing.

1 Introduction

Volatility indices –such as the VIX index– and derivatives written on such indices have gained popularity in markets as tools for hedging volatility risk and as market-based indicators of volatility. Variance swap contracts (see Demeterfi, Derman, Kamal, and Zou (1999)) are increasingly used by market operators to take a pure exposure to volatility or hedge the volatility exposure of options portfolios.

The existence of a liquid market for volatility derivatives such as VIX options, VIX futures and a well developed over-the-counter market for options on variance swaps, and the use of variance swaps and volatility index futures as hedging instruments for other derivatives have led to the need for a pricing framework in which volatility derivatives and derivatives on the underlying asset can be priced in a consistent manner. In order to yield derivative prices in line with their hedging costs, such models should be based on a realistic representation of the joint dynamics of the underlying asset and variance swaps written on this, while also be able to match the observed prices of the liquid derivatives –futures, calls, puts and variance swaps– used as hedging instruments.

In principle, any continuous-time model with stochastic volatility and/or jumps implies some joint dynamics for variance swaps and the underlying asset price. Broadie and Jain (2008a) study the valuation of volatility derivatives in the Heston model; Carr, Geman, Madan, and Yor (2005) study the pricing of volatility derivatives in models based on Lévy processes. However, in many commonly used models the dynamics implied for variance swaps is unrealistic (see Buehler (2006) and Bergomi (2005)): for example, exponential Lévy models imply a constant variance swap term structure while one-factor stochastic volatility models predict perfect correlation among movements in variance swaps at all maturities.

Also, as pointed out in Bergomi (2005, 2008b), the joint dynamics of forward volatilities and the underlying asset is neither explicit nor tractable in most commonly used models, which makes parameter selection/calibration difficult and does not enable the user to choose a parameter to take a view on forward volatility. As a result, classical models such as the Heston model or (time changed) exponential-Lévy models are unable to match empirical properties of

variance swaps and VIX options (see Bergomi (2005, 2008b, 2008a), Buehler (2006), and Gatheral (2008)). Some of these issues can be tackled using multi-factor stochastic volatility models as in Buehler (2006) and Gatheral (2008) but these models remain incapable of reproducing finer features of the data such as the magnitude of the VIX option skew or the different conditional correlations in large and small spot/volatility moves (see Table 1).

A new modeling approach, recently proposed in Bergomi (2005, 2008b, 2008a) (see also related work in Buehler (2006) and Gatheral (2008)) is one in which, instead of modeling “instantaneous” volatility, one models directly the (forward) variance swaps for a discrete tenor of maturities. This approach, which can be seen as the analog of the LIBOR market model for volatility modeling, turns out to be quite flexible and allows to deal with the issues raised above while retaining some tractability.

These models are based on diffusion dynamics where market variables are driven by a multidimensional Brownian motion. Recent price history across most asset classes has pointed to the importance of discontinuities in the evolution of prices; this anecdotal evidence is supplemented by an increasing body of statistical evidence for jumps in price dynamics (e.g. Ait-Sahalia and Jacod (2008), Andersen, Benzoni, and Lund (2002), Barndorff-Nielsen and Shephard (2007), and Cont and Mancini (2007)). Volatility indices such as the VIX have exhibited, especially during the recent crisis, large fluctuations which strongly point to the existence of jumps, or spikes, in volatility. Figure 1, which depicts the daily closing levels of S&P 500 and the VIX volatility index from September 22nd, 2003 to February 27th, 2009, reveals to the simultaneity of large drops in the S&P 500 with spikes in volatility, which corresponds to the well-known “leverage effect”. Comparing the relative changes in the two series (Figure 2) reveals that, while there is already a negative correlation in small changes in the series, large changes –jumps– exhibit an even stronger negative correlation, close to -1 (see Table 1). These observations are confirmed by a recent study in Todorov and Tauchen (2008), who find significant statistical evidence of simultaneous jumps of opposite sign in the VIX and the underlying index. These empirical facts need to be accounted for in a realistic model for variance swap dynamics. They are also important from a pricing perspective: Broadie and Jain (2008b) find that, for a wide range of models and parameter specifications, the effect of discrete sampling on the valuation of volatility derivatives is typi-

cally small while the effect of jumps can be significant. Jumps in volatility are also important in order to produce the positive “skew” of implied volatilities of VIX options (e.g. Bergomi (2008a) and Gatheral (2008)).

1.1 Contribution

Following the approach proposed in Bergomi (2005, 2008b, 2008a), the present work proposes an arbitrage-free modeling framework for the joint dynamics of forward variance swap rates along with the underlying index, which

1. captures the information in index option prices by matching the index implied volatility smiles.
2. is capable of reproducing any observed term structure of variance swap rates.
3. captures the information in options on VIX futures by matching their prices/ implied volatility smiles.
4. implies a realistic joint dynamics of spot and forward implied volatilities, allowing in particular for jumps in volatility and returns.
5. allows for the spot/volatility correlation and the implied volatility skews (of vanilla options) to be parametrized independently.
6. is able to handle the term structure of vanilla skew separately from the term structure of volatility of volatility.
7. is tractable and enables efficient pricing of vanilla options, which is a key point for calibration and implementation of the model.

We address these different issues, while retaining tractability, by introducing a common jump factor which affects variance swaps and the underlying index with opposite signs. Describing these jumps in terms of a Poisson random measure leads to an analytically tractable framework, where VIX options *and* calls/puts on the underlying index can be simultaneously priced using Fourier-based methods. Our framework is in fact a class of models and allows for various specifications of the jump size distribution; we give two worked-out examples and detail their implementation.

The difference between our modeling framework and the Bergomi (2008b) model is mainly the ability to meet points 5), 6) and 7) above. Also, thanks to a semi-analytic representation of call option prices, our model also satisfies the tractability property 1), allowing efficient calibration to the whole implied volatility surface. This is an advantage over the Bergomi (2005, 2008b) model where only the short-term implied volatility smile can be matched.

1.2 Outline

Section 2 describes variance swap contracts, forward variance swap rates and options on variance swaps. The model is described in Section 3 and two model specifications are presented. Section 4 discusses the VIX index and the connection between forward variance swap rates and VIX index futures. Section 5 implements the two specifications of the model and examines their performance in jointly matching VIX options and options on the S&P 500. Section 6 concludes.

2 Variance Swaps and Forward Variances

Consider an underlying asset whose price S is modeled as a stochastic process $(S_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t \geq 0}$ represents the history of the market. We assume the market is arbitrage-free and prices of traded instruments are represented as conditional expectations with respect to an equivalent pricing measure $\mathbb{Q}$. We shall neglect in the sequel corrections due to stochastic interest rates.

2.1 Variance Swaps

The annualized *realized variance* of a (price) process S over a time grid $t = t_0 < \dots < t_k = T$ is given by

$$RV_{t,T} = \frac{M}{k} \sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2, \quad (1)$$

where M is the number of trading days per year.

A variance swap (VS) with maturity T initiated at $t < T$ pays the difference

between the annualized realized variance of the log-returns $RV_{t,T}$ less a strike called the variance swap rate V_t^T , determined such that the contract has zero value at the time of initiation t .

For any semimartingale S , as $\sup_{i=1,\dots,k} |t_i - t_{i-1}| \rightarrow 0$ the realized variance converges to the quadratic variation of the log price:

$$\sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \xrightarrow{\mathbb{Q}} [\log S]_T - [\log S]_t .$$

Approximating the realized variance by the quadratic variation of the log returns is justified when the sampling frequency is daily, as is the case in most variance swap contracts, to warrant replacing the realized variance by its continuous counterpart (see Broadie and Jain (2008b)). The fixed leg paid in a variance swap is then approximated by

$$V_t^T = \frac{1}{T-t} \mathbb{E} ([\log S]_T - [\log S]_t \mid \mathcal{F}_t) , \quad (2)$$

and we refer to this quantity as the (spot) variance swap (VS) rate prevailing at date t for the maturity T .

2.2 Forward Variances

The forward variance, quoted at date t for the period $[T_1, T_2]$ is the strike that sets the value of a forward variance swap running from T_1 to T_2 to zero at time t ; it is given by

$$V_t^{T_1, T_2} = \frac{1}{T_2 - T_1} \mathbb{E} ([\log S]_{T_2} - [\log S]_{T_1} \mid \mathcal{F}_t) \quad (3)$$

$$= \frac{(T_2 - t) V_t^{T_2} - (T_1 - t) V_t^{T_1}}{T_2 - T_1} , \quad (4)$$

where $t < T_1 < T_2$. The last equality follows easily by substitution of (2). In particular, as noted in Bergomi (2005, 2008a) forward variances have the martingale property under the pricing measure. Choosing $t < s < T_1 < T_2$ it follows by substitution of (3) and the use of the law of iterated expectations that

$$\mathbb{E} (V_s^{T_1, T_2} \mid \mathcal{F}_t) = V_t^{T_1, T_2} \quad (5)$$

which shows that forward variances are martingales under the pricing measure $\mathbb{Q}$.

Assume that a set of settlement dates is given

$$T_0 < T_1 < \dots < T_n$$

known as the tenor structure and that the interval between two tenor dates is fixed, $\tau_i = T_{i+1} - T_i = \tau$ (equal to 30 days if we use the tenor structure of the VIX futures). We define the *forward variance* over the time interval $[T_i, T_{i+1}]$ as

$$V_t^i = V_t^{T_i, T_{i+1}}. \quad (6)$$

An example of a forward variance swap term structure is given in Figure 3.

2.3 Options on Forward Variance Swaps

A call option with strike K and maturity T_1 on a forward variance swap for the period $[T_1, T_2]$ gives the holder the option to enter at date T_1 into a variance swap running from T_1 to T_2 with some predetermined strike K . Hence, the value at time T_1 is

$$\begin{aligned} \pi(T_1, T_1, T_2, K) &= \left(e^{-\int_{T_1}^{T_2} r_s ds} \mathbb{E}(RV_{T_1, T_2} - K \mid \mathcal{F}_{T_1}) \right)^+ \\ &= e^{-\int_{T_1}^{T_2} r_s ds} \left(V_{T_1}^{T_1, T_2} - K \right)^+, \end{aligned} \quad (7)$$

which gives us the time t value

$$\pi(t, T_1, T_2, K) = e^{-\int_t^{T_2} r_s ds} \mathbb{E} \left(\left(V_{T_1}^{T_1, T_2} - K \right)^+ \mid \mathcal{F}_t \right). \quad (8)$$

From this expression it is clear that having tractable dynamics for the forward variance swap rates enable the pricing of options on variance swaps. It should be noted here that at the present the market for options on variance swaps is illiquid. On the contrary the related market for options on VIX futures is highly liquid and the model will be calibrated to this. The VIX index is discussed in Section 4.

3 A Model for the Joint Dynamics of Variance Swaps and the Underlying Index

Our goal is to construct a model which

- allows for an arbitrary initial variance swap term structure.
- allows to specify directly the dynamics of the observable forward variance swaps V_t^i on a discrete tenor of maturities $(T_i, i = 1..n)$.
- allows for flexible modeling of the variance swap curve: for example, it should be able to accommodate the fact that variance swaps at the long end of the maturity curve have a lower variability than those at the short end.
- allows for jumps in volatility (see Todorov and Tauchen (2008)) and in the underlying asset (see Ait-Sahalia and Jacod (2008), Andersen, Benzoni, and Lund (2002), Barndorff-Nielsen and Shephard (2007), and Cont and Mancini (2007)).
- allows for a flexible specification of the spot price dynamics.
- is analytically tractable i.e. leads to efficient numerical methods for pricing/calibration of calls/puts both on the underlying and volatility derivatives such as options on forward variance swaps and options on forward volatility.

In order to achieve these goals, we first specify the dynamics of (a discrete tenor of) forward variance swaps (Section 3.1) using an affine specification which allows Fourier-based pricing of European-type volatility derivatives. Once the dynamics of forward variance swaps has been fixed, we specify a jump-diffusion dynamics for the underlying asset which is compatible with the variance swap dynamics (Section 3.2). Presence of a jump component as well as a diffusion component in the underlying asset allows us to satisfy this compatibility condition while simultaneously matching values/implied volatilities of options on the underlying asset (Section 3.3).

3.1 Variance Swap Dynamics

Given that the forward variance swap rate is a (positive) martingale under the pricing measure, we model it as

$$\begin{aligned} V_t^i &= V_0^i e^{X_t^i} \\ &= V_0^i \exp \left\{ \int_0^t \mu_s^i ds + \int_0^t \omega e^{-k_1(T_i-s)} dZ_s + \int_0^t \int_{\mathbb{R}} e^{-k_2(T_i-s)} x J(dx ds) \right\}, \end{aligned} \quad (9)$$

where $J(dxdt)$ is a Poisson random measure with compensator $\nu(dx)dt$ and Z a Wiener process independent of J . The martingale condition imposes

$$\mu_t^i = -\frac{1}{2}\omega^2 e^{-2k_1(T_i-t)} - \int_{\mathbb{R}} (\exp\{e^{-k_2(T_i-t)}x\} - 1) \nu(dx) . \quad (10)$$

For $t > T_i$ we let $V_t^i = V_{T_i}^i$. In the case of finite jump intensity the expression (9) reduces to

$$V_t^i = V_0^i \exp \left\{ \int_0^t \mu_s^i ds + \int_0^t \omega e^{-k_1(T_i-s)} dZ_s + \sum_{j=0}^{N_t} e^{-k_2(T_i-\tau_j)} Y_j \right\}, \quad (11)$$

where N is a Poisson process with intensity λ , τ_j its jump times, and Y_j IID random variables with distribution F independent of Z . In this case the Lévy measure has the form $\nu(dx) = \lambda F(dx)$.

This specification allows for jumps and the exponential functions inside the integrals allow to control the term structure of volatility of volatility. For example, if k_1 is large and k_2 small, the diffusion Z results mostly in fluctuations at the short end of the curve, while the jumps impact the entire variance swap curve. Correlated Brownian factors can be added if more flexibility in the variance swap curve dynamics is desired, e.g. correlation between movements in the short and the long end of the curve. Likewise, functional forms other than exponentials can be used in (9), although the exponential specification is quite flexible as we will observe in the examples.

Expressions such as (8) can be evaluated using Fourier-based methods (e.g. Carr and Madan (1999), Lewis (2001), and Cont and Tankov (2004)) given the

characteristic function of $X_{T_i}^i$, which in this case has a simple form:

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] = \exp \left\{ -\frac{1}{2}u^2 \int_0^{T_i} \omega^2 e^{-2k_1(T_i-s)} ds + iu \int_0^{T_i} \mu_s^i ds \right. \\ \left. + \int_0^{T_i} \int_{\mathbb{R}} (\exp \{ iue^{-k_2(T_i-s)}x \} - 1) \nu(dx) dt \right\} . \end{aligned} \quad (12)$$

3.2 Dynamics of the Underlying Asset

Once the dynamics of forward variance swaps V_t^i for a discrete set of maturities $T_i, i = 1..n$ has been specified, this imposes some constraints on the (risk neutral) dynamics of the underlying asset $(S_t)_{t \geq 0}$. S should be such that

1. the discounted spot price $\hat{S}_t = \exp(-\int_0^t (r_s - q_s) ds) S_t$ is a (positive) martingale, where q_t is the dividend yield.
2. the dynamics of the spot price is compatible with the specification of V_t^i , which puts the following constraint on the quadratic variation process $[\log S]$ of the log-price:

$$\frac{1}{T_{i+1} - T_i} \mathbb{E} [[\log S]_{T_{i+1}} - [\log S]_{T_i} | \mathcal{F}_t] = V_t^i \quad (13)$$

To these constraints we add a third requirement, namely that:

3. the model values of calls/puts on S match the observed prices across strikes and maturities.

Typically we need at least two distinct parameters/degrees of freedom in the dynamics of the underlying asset in order to accommodate points 2) and 3) above.

The models in Bergomi (2005, 2008b) propose to achieve this by introducing a random "local volatility" function which is reset at each tenor date and chosen such at time T_i to match the observed value of $V_{T_i}^i$. This procedure guarantees coherence between the variance swaps and the underlying asset dynamics but leads to a loss of tractability: even vanilla call options need to be priced by Monte Carlo simulation for maturities $T > T_1$.

We adopt here a different approach which allows a greater tractability while simultaneously allowing for jumps in the volatility and the price. In fact, as we will see, introducing jumps *is* the key to tractability.

The underlying asset is driven by

- a Brownian motion W , correlated with the diffusion component Z driving the variance swaps: $\langle W, Z \rangle_t = \rho t$.
- a jump component, which is driven by the *same* Poisson random measure J which drives jumps in the variance. However, we allow different jump amplitudes in the underlying and the forward variance.

Presence of a jump component as well as a diffusion component in the underlying asset allows us to satisfy this compatibility condition while simultaneously matching values/implied volatilities of options on the underlying asset (Section 3.3).

For each interval $[T_i, T_{i+1})$ we introduce a stochastic diffusion coefficient σ_i and a function $u_i(x, V^i)$ which expresses the size of the jump in the underlying asset in terms of the jump size x in forward variance level V^i and V^i itself. We will give in Section 3.4 some simple and flexible specifications for this function u_i but most developments below hold for arbitrary choice of u_i .

We will present here the detailed computations in the (typically useful) case where $T = T_m$, for $m = 1, \dots, n$, but the analysis can be easily generalized to any maturity. The dynamics of the underlying asset is then specified as

$$S_{T_m} = S_0 \exp \left\{ \int_0^{T_m} (r_s - q_s) ds + \sum_{i=0}^{m-1} \mu_i (T_{i+1} - T_i) + \sigma_i (W_{T_{i+1}} - W_{T_i}) + \sum_{i=0}^{m-1} \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\}, \quad (14)$$

where $\mu_i = -\frac{1}{2}\sigma_i^2 - \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx)$ and the σ_i s are stochastic and revealed at time T_i by relation (18) below to match the known variance swap value $V_{T_i}^i$ at this time point. The drift term μ_i is also stochastic and $\mathcal{F}_{T_i}$ -measurable. The random jump measure J in the index dynamics is the same as that in the VS dynamics, so the index and the variance swaps jump simultaneously. u_i is a deterministic function of x and $V_{T_i}^i$ chosen to match the observed volatility smiles implied by prices of options on the spot. W is correlated with Z with a correlation ρ .

When the jump intensity is finite expression (14) reduces to

$$S_{T_m} = S_0 \exp \left\{ \int_0^{T_m} (r_s - q_s) ds + \sum_{i=0}^{m-1} \mu_i (T_{i+1} - T_i) + \sigma_i (W_{T_{i+1}} - W_{T_i}) + \sum_{i=0}^{m-1} \sum_{T_i \leq \tau_j < T_{i+1}} u_i (Y_j, V_{T_i}^i) \right\}. \quad (15)$$

As far as pricing of vanilla instruments is concerned, the model can be viewed simply as a flexible (and analytically tractable) parameterization of the joint distribution of forward variance swap rates and the underlying asset on a set of tenor dates. At this level the only assumption we are making is that this joint distribution is infinitely divisible (see Sato (1999)) and our model is just a flexible parameterization of the Lévy triplet of the distribution. This makes it particularly easy to price any payoff which involves these variables only at tenor dates.

The assumption of a common jump factor which affects the variance swaps and the underlying in opposite directions is not only analytically convenient but quite realistic from an empirical perspective. As revealed in Table 1, while the unconditional correlation of daily returns of the VIX index and the S&P 500 from September 22nd, 2003 to February 27th, 2009 is -74% , the *conditional* correlation between the two sub-series for daily moves in the S&P 500 less than 0.5% drops to -0.45 while the conditional correlation of moves greater than 5% , which can be interpreted as jumps, is -93% , close to -100% . This observation is also in agreement with the findings reported in Todorov and Tauchen (2008) using non-parametric methods. This feature, which has no equivalent in diffusion-based stochastic volatility models such as Bergomi (2008b) and Gatheral (2008), is a generic property of our framework.

Given the dynamics (14) of the underlying asset, the quadratic variation over the time interval $[T_0, T_m]$ is given by

$$[\log S]_{T_m} - [\log S]_{T_0} = \sum_{i=0}^{m-1} \sigma_i^2 (T_{i+1} - T_i) + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 J(ds dx). \quad (16)$$

By taking expectation on (16) the variance swap rate in (2) equals

$$V_{T_0}^{T_m} = \sum_{i=0}^{m-1} \frac{T_{i+1} - T_i}{T_m - T_0} \left(\mathbb{E} [\sigma_i^2 \mid \mathcal{F}_{T_0}] + \mathbb{E} \left[\int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx) \mid \mathcal{F}_{T_0} \right] \right),$$

and the forward variance at time t for the interval $[T_i, T_{i+1}]$ equals

$$V_t^i = \mathbb{E} [\sigma_i^2 \mid \mathcal{F}_t] + \mathbb{E} \left[\int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx) \mid \mathcal{F}_t \right]. \quad (17)$$

Equation (17) has to hold at all times, but since V_t^i is a martingale we just need to ensure that at time T_i

$$V_{T_i}^i = \sigma_i^2 + \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx). \quad (18)$$

Having fixed $\nu(dx)$ it is seen that any observed forward variance structure can be matched by an appropriate choice of the σ_i s which leaves the parameters in u_i free to calibrate to option prices.

3.3 Pricing of Vanilla Options

Let us now explain the procedure for pricing European calls and puts on S in this framework. The aim is to compute in an efficient manner the value of call options of various strikes and maturities at, some initial date $t = 0$:

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds} \mathbb{E}[(S_{T_m} - K)^+ \mid \mathcal{F}_0]. \quad (19)$$

Denote by $\mathcal{F}_t^{(Z, J)}$ the filtration generated by the Wiener process Z and the Poisson random measure J . By first conditioning on the factors driving the variance swap curve and using the iterated expectation property

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds} \mathbb{E}[\mathbb{E}[(S_{T_m} - K)^+ \mid \mathcal{F}_{T_m}^{(Z, J)}] \mid \mathcal{F}_0] \quad (20)$$

we obtain a *mixing formula* for valuing call options:

Proposition 1 *The value $C(0, S_0, K, T_m)$ of a European call option with maturity T_m and strike K is given by*

$$C(0, S_0, K, T_m) = \mathbb{E}[C^{BS}(U_{T_m}, K, T_m; \sigma_*)], \quad (21)$$

where $C^{BS}(S, K, T; \sigma)$ denotes the Black-Scholes formula for a call option with strike K and maturity T :

$$\sigma_*^2 = \frac{1}{T_m} \sum_{i=0}^{m-1} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i), \quad (22)$$

and U_{T_m} is a $\mathcal{F}_{T_m}^{(Z, J)}$ -measurable random variable given by

$$U_{T_m} = S_0 \exp \left\{ \sum_{i=0}^{m-1} - \left(\frac{1}{2} \sigma_i^2 \rho^2 + \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx) \right) (T_{i+1} - T_i) \right. \\ \left. + \rho (Z_{T_{i+1}} - Z_{T_i}) \sigma_i + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\} \quad (23)$$

and the expectation in (21) is taken with respect to the law of (Z, J) .

Proof. Conditional on $\mathcal{F}_{T_m}^{(Z, J)}$, the increments of Z and the paths of the variance swap rates, and in particular $V_{T_i}^i$, thus the corresponding σ_i are known for $i = 0, 1, \dots, m$ from equation (18). Moreover

$$W_{T_{i+1}} - W_{T_i} \stackrel{d}{=} \rho (Z_{T_{i+1}} - Z_{T_i}) + \sqrt{(1 - \rho^2)} (\hat{W}_{T_{i+1}} - \hat{W}_{T_i}),$$

where the Wiener processes Z and $\hat{W}$ are independent. The process S_t can therefore be decomposed as

$$S_{T_m} = U_{T_m} \exp \left\{ \int_0^{T_m} (r_s - q_s) ds + \sum_{i=0}^{m-1} \left(-\frac{1}{2} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i) \right. \right. \\ \left. \left. + \sigma_i \sqrt{1 - \rho^2} (\hat{W}_{T_{i+1}} - \hat{W}_{T_i}) \right) \right\},$$

where $\sigma_i^2 = V_{T_i}^i - \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx)$ and

$$U_{T_m} = S_0 \exp \left\{ \sum_{i=0}^{m-1} - \left(\frac{1}{2} \sigma_i^2 \rho^2 + \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx) \right) (T_{i+1} - T_i) \right. \\ \left. \rho (Z_{T_{i+1}} - Z_{T_i}) \sigma_i + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\}$$

is $\mathcal{F}_{T_m}^{(Z,J)}$ -measurable. In distribution we have

$$S_{T_m} \stackrel{d}{=} U_{T_m} e^{\int_0^{T_m} (r_s - q_s) ds + \mu_* T_m + \sigma_* \tilde{W}_{T_m}},$$

where σ_* is given in (22) and

$$\mu_* = \frac{1}{T_m} \sum_{i=0}^{m-1} -\frac{1}{2} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i), \quad (24)$$

where we notice that $\mu_* = -\frac{1}{2} (\sigma_*)^2$. Hence, given $\mathcal{F}_{T_m}^{(Z,J)}$, the inner conditional expectation in (20) reduces to the evaluation of the Black-Scholes formula

$$\mathbb{E}[(S_{T_m} - K)_+ | \mathcal{F}_{T_m}^{(Z,J)}] = C^{BS}(U_{T_m}, K, T_m; \sigma_*),$$

where σ_* depends on all the u_i s through (18) and (22). ■

This result has interesting consequences for pricing and calibration of vanilla contracts. Note that the outer expectation can be computed by Monte Carlo simulation of the Z and J : with N simulated sample paths for Z and J we obtain the following approximation

$$\hat{C}_N = \frac{1}{N} \sum_{k=1}^N C^{BS}(U_{T_m}^{(k)}, K, T_m; \sigma_*(k)) \xrightarrow{N \rightarrow \infty} C(0, S_0, K, T_m). \quad (25)$$

Since the averaging is done over the variance swap factors Z and J , this is a deterministic function of the parameters in the u_i s. This will prove useful when calibrating the model using option data, since we *do not* have to run the N Monte Carlo simulations for each calibration trial.

Equation (25) thus allows to calibrate the model to the *entire* implied volatility surface in an efficient manner, in contrast to the Bergomi model where it is only possible to calibrate to at-the-money slopes of the implied

volatilities (ATM skews).

3.4 Examples

Different classes of models in the above framework can be obtained by various parameterizations of the Lévy measure $\nu(dx)$ describing the jumps and for the functions u_i appearing in (14). In this paper we implement two model parameterizations; normal jumps and double exponential jumps in the variance swaps and the index.

3.4.1 Normally Distributed Jumps

In the first example, we specify the Lévy measure as

$$\nu(dx) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}} dx, \quad (26)$$

where m and δ^2 is the mean and variance in a normal distribution and λ the intensity of the jumps. With $\nu(dx)$ specified the characteristic function in (12) can be calculated (see (43) in Appendix A) and prices of options on the variance swaps can be computed by Fourier transform methods.

We now consider the following specification for u_i , which relates the jumps in the variance swap to the jumps in the underlying asset

$$u_i(x, V_{T_i}^i) = \left(\frac{V_{T_i}^i}{V_0^i} \right)^{\frac{1}{2}} b_i x, \quad (27)$$

but any functional form can be used as long as (18) leads to positive values for σ_i .

This gives us the σ_i s at time T_i

$$\begin{aligned} \sigma_i^2 &= V_{T_i}^i - \frac{V_{T_i}^i}{V_0^i} \int_{\mathbb{R}} (b_i x)^2 \nu(dx) \\ &= V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i} (b_i^2 m^2 + b_i^2 \delta^2). \end{aligned}$$

In order to achieve non-negative values for σ_i^2 we require

$$\lambda (b_i^2 m^2 + b_i^2 \delta^2) \leq V_0^i. \quad (28)$$

3.4.2 Exponentially Distributed Jumps

To introduce an asymmetry in the tail of the jump size distribution we can use a double exponential distribution introduced in Kou (2002):

$$\nu(dx) = \lambda \left(p\alpha_+ e^{-\alpha_+ x} \mathbf{1}_{x \geq 0} + (1-p) \alpha_- e^{-\alpha_- |x|} \mathbf{1}_{x < 0} \right) dx, \quad (29)$$

where p denote the probability of a positive jump, $1/\alpha_+$ and $1/\alpha_-$ the expected positive and negative jump sizes, respectively. In this specification the characteristic function for the forward variance swap is given by (45).

As above, we specify the u_i function as in (27), which gives rise to the following constraint on the σ_i s

$$\sigma_i^2 = V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i} \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)b_i^2}{\alpha_-^2} \right).$$

To ensure positive σ_i s we constrain the calibration by

$$\lambda \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)b_i^2}{\alpha_-^2} \right) \leq V_0^i. \quad (30)$$

The parameters b_i in the above specifications can take any value as long as (28) respectively (30) are satisfied. We will see later, how the calibration of b_i entails that jumps in the underlying have opposite direction to the jumps in the VIX futures. Together with negative correlation between Z and W , this feature enables the model to generate positive skews for the implied volatility of VIX options and negative skews for the implied volatility of calls and puts on the underlying index, as observed in empirical data. This property is achieved without the need to add extra volatility factors, as proposed recently in e.g. Buehler (2006), Duffie, Pan, and Singleton (2000), and Gatheral (2008) where multi-factor mean-reverting dynamics on volatility and volatility of volatility are imposed to accommodate this behavior, resulting in a loss of tractability.

3.5 Impact of Jumps on the Valuation of Variance Swaps

As noted above, recent price history across most asset classes has pointed to the importance of discontinuities in the evolution of prices; this is supplemented by an increasing body of statistical evidence for jumps in price dynamics (Ait-

Sahalia and Jacod (2008), Andersen, Benzoni, and Lund (2002), Barndorff-Nielsen and Shephard (2007), and Cont and Mancini (2007)). Before moving on we would like to digress on the effect of jumps in the underlying index dynamics when pricing variance swaps.

As noted in Neuberger (1994), if the underlying asset follows continuous dynamics, its quadratic variation and hence also the payoff from a variance swap can be replicated by continuous rebalancing of a position in the underlying asset and a static position in a log contract, which in turn can be replicated by a static portfolio of calls and puts (shown in Carr and Madan (1998) and Demeterfi, Derman, Kamal, and Zou (1999)):

$$\frac{1}{T-t} ([\log S]_T - [\log S]_t) = \frac{2}{T-t} \left(\int_t^T \frac{dS_t}{S_t} - \log \frac{S_T}{S_t} \right)$$

and then under discrete sampling

$$\frac{M}{k} \sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 = 2 \frac{M}{k} \sum_{i=1}^k \left(\frac{\Delta S_i}{S_{i-1}} - \log \frac{S_i}{S_{i-1}} \right).$$

The left hand side is the realized variance paid and the right hand side is the payoff from the hedging strategy. Figure 4 depicts the day to day profit-loss from holding a long position in the replicating portfolio and a short position in the three month variance swap. Variance Point (VP) is defined as realized variance multiplied by 10,000. As can be seen, the replicating portfolio performs relatively well most of the days, but days in which the index moved considerably show an error of up to -1.5 VP. Table 2 shows the error from holding the portfolio relative to the realized variance over the three month period. The maximum error from following this strategy arises in the June 2008 - September 2008 variance swap and results in an error of -0.243% . These four real examples indicate that the error from assuming continuous dynamics have been historically small.

Sometimes this is taken as an argument that jumps are not significant when variance swaps are priced. This can very well be a false conclusion, which can be realized from the following. When assuming continuous dynamics, the error that arises when valuing variance swap rates is given by (see Carr and Wu

(2006))

$$-\frac{2}{T-t}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}}\left(e^y-1-y-\frac{y^2}{2}\right)\nu(dy)ds\mid\mathcal{F}_t\right],$$

where y is the jump size in the index. That this expectation is small under the physical measure does not entail that the mean of the error under the *pricing* measure should be zero. This common fallacy is akin to saying that since crashes occur infrequently deep out-of-the-money puts should be priced at zero: recent experience shows that such mistaken assertions can be costly. In fact adding jumps with negative mean value to the log-price dynamics should increase the value of the variance swap rate, since being long a variance swap can be seen as an insurance against downward movements in the underlying asset and therefore have a risk premium attached to it (see Carr and Wu (2009)).

4 The VIX Index

As mentioned, the market for options on the VIX index is well developed, and unless data on options on variance swaps is available, information stemming from this market should be used when the dynamics of the variance swaps are specified. For completeness we describe the VIX index in the first subsection, but the reader is referred to CBOE (2003) and Carr and Wu (2006) for a detailed discussion. In the second subsection we derive our choice for the VIX futures dynamics given the dynamics of the forward variance swaps .

4.1 Description

The Chicago Board Options Exchange (CBOE) introduced the volatility index, VIX, in 1993, and it has since become the industry benchmark for market volatility. The VIX index provides investors with a quote on the expected market volatility over the next 30 calendar days. In September 2003 the VIX was revised such that the index is now calculated on the basis of put and call options on S&P 500 instead of the S&P 100 index. Furthermore, the index was changed to calculate the expected volatility on the basis of options in a wide range of strike prices, where the original VIX index was based on at-the-money strikes only.

The general formula used in the VIX calculation at time t is

$$VS_t^T = \frac{2}{T-t} \sum_i \frac{\Delta K_i}{K_i^2} e^{r_t^T(T-t)} Q(K_i, T; t) - \frac{1}{T-t} \left(\frac{F_t}{K_0} - 1 \right)^2, \quad (31)$$

where T is time of expiration, K_i the strike price of the i th out-of-the money option, ΔK_i the resolution of the strike grid:

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}.$$

For the lowest strike, ΔK_i is the difference between the lowest and second lowest strikes. This also applies to the highest strike. r_t^T is the risk-free interest rate to expiration, $Q(K_i, T; t)$ the midpoint between the bid and ask prices on the option with strike K_i and maturity T , either a call if $K_i > F_t$ or a put if $K_i < F_t$, where F_t is the forward S&P 500 index price. K_0 is the first strike below the forward index level F_t . Normally, no options will expire in exactly 30 days. Therefore, CBOE interpolates between $VS_t^{T_1}$ and $VS_t^{T_2}$

$$VIX_t = 100 \sqrt{\frac{365}{30} \left((T_1 - t) VS_t^{T_1} \frac{N_{T_2} - 30}{N_{T_2} - N_{T_1}} + (T_2 - t) VS_t^{T_2} \frac{30 - N_{T_1}}{N_{T_2} - N_{T_1}} \right)}, \quad (32)$$

where N_{T_1} and N_{T_2} denote the actual number of days to expiration for the two options maturities, $N_{T_1} < 30 < N_{T_2}$.

The VIX is often presented as an indicator of the expected volatility over the next 30 days. It is not immediately clear from either (31) or (32) that this is in fact the case. For notational simplicity assume that the jumps exhibit finite variation. Then it can be shown that the realized variance of returns can be decomposed into three components

$$\begin{aligned} [\log S]_T - [\log S]_t = & \frac{2}{T-t} \left(\int_0^{F_t} \frac{1}{K^2} (K - S_T)^+ dK \right. \\ & + \int_{F_t}^\infty \frac{1}{K^2} (S_T - K)^+ dK + \int_t^T \left(\frac{1}{F_{s-}} - \frac{1}{F_t} \right) dF_s \\ & \left. - \int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) J(dy ds) \right), \end{aligned} \quad (33)$$

where J is a Poisson random measure with Lévy measure $\nu(dx)$, driving the

jumps in the underlying asset. Notice how the realized variance can be replicated by trading in options and futures up to a discontinuous jump component.

Taking expectations with respect to the pricing measure in (33) we obtain

$$V_t^T = \frac{2}{T-t} e^{\int_t^T r_s ds} \int_0^\infty \frac{Q(K, T; t)}{K^2} dK - \frac{2}{T-t} \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) \nu(dy) ds \mid \mathcal{F}_t \right]. \quad (34)$$

Equation (31) is a discretization of the first term in (34). The extra term $\frac{1}{T-t} \left(\frac{F_t}{K_0} - 1 \right)^2$ in (31) is a contribution due to the discretization around F_t .

The square of the VIX index is thus a model free estimate of the expected volatility over the next 30 days given continuous dynamics of the underlying but under general price processes the error term is given by the last term in (34):

$$VIX_t^2 = V_t^{t+30\text{days}} + \frac{2}{T-t} \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) \nu(dy) ds \mid \mathcal{F}_t \right]. \quad (35)$$

4.2 VIX Futures

Let VIX_t^i denote the VIX futures price for the interval $[T_i, T_{i+1}]$ seen at time t . For $t = T_i$ we have

$$(VIX_{T_i}^i)^2 = V_{T_i}^i + 2 \int_{\mathbb{R}} (e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2}) \nu(dx).$$

We also have the martingale property for $t < T_i$

$$VIX_t^i = \mathbb{E} [VIX_{T_i}^i \mid \mathcal{F}_t]$$

By Jensen's inequality for convex functions

$$\begin{aligned} (VIX_t^i)^2 &\leq \mathbb{E} \left[(VIX_{T_i}^i)^2 \mid \mathcal{F}_t \right] \\ &= V_t^i + 2 \mathbb{E} \left[\int_{\mathbb{R}} e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2} \nu(dx) \mid \mathcal{F}_t \right], \end{aligned} \quad (36)$$

since V_t^i is a martingale. Equation (36) shows that there is a “convexity correction” connecting VIX futures to forward variance swap rates.

To obtain a tractable framework, we first characterize the dynamics of $\sqrt{V_t^i}$ then propose an approximation which has a closed-form characteristic function.

Proposition 2 *The process $\sqrt{V_t^i}$ has the multiplicative decomposition*

$$\sqrt{V_t^i} = \sqrt{V_0^i} M_t^i A_t^i$$

where A_t^i is a finite variation process and M_t^i is an exponential martingale given by

$$M_t^i = \exp \left\{ \int_0^t \eta_s^i ds + \frac{1}{2} \int_0^t \omega e^{-k_1(T_i-s)} dZ_s + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx ds) \right\}, \quad (37)$$

where

$$\eta_t^i = -\frac{1}{8} \omega^2 e^{-2k_1(T_i-t)} - \int_{\mathbb{R}} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-t)} x \right\} - 1 \right) \nu(dx). \quad (38)$$

The proof of this result is given in Appendix B.

Using this result we approximate VIX_t^i with

$$VIX_t^i \simeq VIX_0^i M_t^i = VIX_0^i \exp(Y_t^i). \quad (39)$$

where $Y_t^i = \ln M_t^i$. This approximation leaves the initial level of the VIX^i and the volatility of VIX^i untouched and thus is relevant for pricing VIX derivatives. As in the case of the forward variance swaps, once the Lévy measure $\nu(dx)$ is specified, the characteristic function for the exponent in (39) can be easily computed, so options on the VIX index can be priced by Fourier transform methods.¹

¹The characteristic functions for the specifications in Section 3.4 are given in Appendix A.

5 Implementation

In this section the model specifications described in Section 3.4 are implemented on data on the VIX and the S&P 500 implied volatility smiles.

5.1 Data

We assess the performance of the model using prices from August 20th, 2008 on a range of VIX put and call options for five maturities, VIX futures for the same maturities, the dividend yield on S&P 500, call and put options on S&P 500 for six maturities and for the various maturities we also have the corresponding prices of the futures on S&P 500, from which we also derive a discount curve. We also observe 3 month forward variance swap rates for various maturities. The variance swap rates have been converted to forward 1 month variance swap rates by simple linear extrapolation and these are depicted in Figure 3. Options for which the bid price is zero were removed.

5.2 Calibration

The calibration of the model consists of three steps:

1. Determine the parameters controlling the variance swap dynamics by calibration to VIX options using the characteristic function in (44) or (46) and Fourier transform methods.
2. Use the parameters from the first step to simulate N paths of the variance swaps and store the increments of Z , the jump times and jump sizes along with the $V_{T_i}^i$ s.
3. Calibrate to options on the index recursively by use of (25).

In the first step we compute model prices by the fast Fourier transform technique described in Carr and Madan (1999) using the revised method in Cont and Tankov (2004). The calibration is performed by minimizing the sum of squared errors weighted by the inverse bid-ask spread across all maturities and strikes on out-of-the-money options on the VIX futures

$$SE = \sum_{\text{options}} \frac{1}{Q_{Ask} - Q_{Bid}} (Q_{Market,Mid} - Q_{Model})^2, \quad (40)$$

using a gradient-based minimization algorithm.

The resulting parameters for the two specifications are shown in Table 3 and it also includes the resulting average relative percentage bid-ask error from the calibration,

$$\text{Error} = \frac{1}{\#\{\text{options}\}} \sum_{\text{options}} \frac{\max\{(Q_{Model} - Q_{Ask})^+, (Q_{Bid} - Q_{Model})^+\}}{Q_{Market, Mid}}. \quad (41)$$

We see how both model specifications are able to achieve very low calibration error. Note the high value of k_1 compared to k_2 , which implies (in the risk-neutral dynamics) that the diffusion component Z is mainly causing fluctuations at the short end of the variance swap curve, while the jumps impact the entire curve.

Figures 5 and 6 depict the (Black) implied volatility for the bid, ask, mid and model prices of VIX options as a function of moneyness. We observe that model prices fit very well within the bid-ask spread for almost all the observations in both specifications. This performance should be compared with flat model implied volatilities in the Bergomi (2005) model and downward sloping volatilities in the Heston (1993) model. The Bergomi (2008b) model is able to generate positive skews on volatility of volatility via the use of a Markov functional mapping for the forward variances, but it should be noted that the model can only price vanilla options on the underlying index by Monte Carlo simulation.

Step 2 is done by discretization of (9). In this example we have used $2 \cdot 10^5$ simulated sample paths. Simulation of the jumps is straightforward, since they arrive at constant intensity and the jump sizes are computed by draws from a normal/exponential distribution times the scaling $e^{-k_2(T_i - \tau)}$, where τ is the jump time. For details on how to simulate the increments in the variance swaps due to the diffusion the reader is referred to Bergomi (2005).

In step 3 we have fixed the correlation parameter between the continuous components to $\rho = -0.45$.² Then, the last step is implemented step-wise by matching the model prices (25) to market prices on the shortest maturity out-of-the-money options. Again, this is achieved by minimizing the SE (40).

²Instead one could also specify a correlation ρ_i on each interval and include that parameter in the calibration for each interval. We tried this, but the performances of the models did not improve with this added flexibility.

This yields b_0 . Then b_1 are found in the same way by calibrating to the next maturing options by the use of b_0 for the first interval in (23). The remaining b_i s are estimated in a similar manner.

The estimated parameters for the two different specifications of the jump size distributions are shown in Table 4 along with the pricing error in (41) for each maturity. The mean and standard deviations of the jumps before scaling with $(V_{T_i}^i/V_0^i)^{\frac{1}{2}}$ are also included. The values of b_i for $i = 0, \dots, 3$ relate to the monthly distributions of the index up to the maturity of the options expiring on December 19th, 2008. For the last two option maturities the interval between the expirations are three months and hence b_i for $i = 4, 5$ relate to the three month distributions of the index. Double exponential jumps perform slightly better on the first interval, but there is not a strong difference between the two specifications, showing that the model performance is not very sensitive to the choice of the jump size distribution.

Figures 7 and 8 show the result of the calibration to the S&P 500 options. The fit of the two specifications are practically indistinguishable and overall the calibrations perform well for both with somewhat less success on the shortest maturity.

We conclude this section with an examination of the size of the jump error term when valuing the forward variance swap rates V^i under the assumption of continuous dynamics given the models implemented in this paper are “true”. Recall that the error is given by

$$\varepsilon_i = -2\mathbb{E} \left[\int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2} \right) \nu(dx) \mid \mathcal{F}_0 \right]. \quad (42)$$

In Table 5 the absolute errors implied by the calibration of the two model specifications are computed for each interval. Moreover, the errors relative to the initial forward variance swap value are reported. For the first interval it is 2.2%/2.3% and then it roughly increases as i increases. The reported errors for the last two intervals are taken as the average of the three monthly errors relative to the average of the three monthly forward variances covering the same intervals.

6 Conclusion

We have presented a model for the joint dynamics of a set of forward variance swap rates along with the underlying index. Using Lévy processes as building blocks, this model leads to a tractable pricing framework for variance swaps, VIX futures and vanilla call/put options, which makes calibration of the model to such instruments feasible. This tractability feature distinguishes our model from previous attempts in e.g. Bergomi (2005, 2004) and Gatheral (2008) which only allow for full Monte Carlo pricing of vanilla options.

Our model reproduces salient empirical features of variance swap dynamics, in particular the strong negative correlation of large index moves with large moves in the VIX and the positive skew observed in implied volatilities of VIX options, by introducing a common jump component in the variance swaps and the underlying asset. Using two different specifications for the jump size distribution (Lévy measure) we have illustrated the feasibility of the numerical implementation, as well as the capacity of the model to match a complete set of market prices of vanilla options and options on the VIX. Our model can be used to price and hedge various payoffs sensitive to forward volatility, such as cliquet or forward start options, as well as volatility derivatives, in a manner consistent with market prices of simpler instruments such as calls, puts or variance swaps which are typically used for hedging them.

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A Characteristic Functions

A.1 Normally Distributed Jumps

Variance Swaps

We have from Sato (1999) that the characteristic function of $X_{T_i}^i$ is given by

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] = \exp \left\{ -\frac{1}{2}u^2 \int_0^{T_i} \omega^2 e^{-2k_1(T_i-s)} ds + iu \int_0^{T_i} \mu_s^i ds \right. \\ \left. + \int_0^{T_i} \int_{\mathbb{R}} (\exp \{ iue^{-k_2(T_i-s)} x \} - 1) \nu(dx) dt \right\}. \end{aligned}$$

Notice now that

$$\int_0^{T_i} e^{-2k_1(T_i-s)} ds = \frac{1 - e^{-2k_1 T_i}}{2k_1}$$

and

$$\mathbb{E} [e^Y] = e^{m + \frac{\delta^2}{2}}$$

for $Y \sim N(m, \delta^2)$. Inserting this into the expression for the characteristic function we arrive at

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] = \exp \left\{ -\frac{1}{2}\omega^2 iu \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{2}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ \left. - iu\lambda \int_0^{T_i} \left(\exp \left\{ e^{-k_2(T_i-s)} m + \frac{1}{2}e^{-2k_2(T_i-s)} \delta^2 \right\} - 1 \right) ds \right. \\ \left. + \lambda \int_0^{T_i} \left(\exp \left\{ iue^{-k_2(T_i-s)} m - \frac{1}{2}u^2 e^{-2k_2(T_i-s)} \delta^2 \right\} - 1 \right) ds \right\}. \end{aligned} \quad (43)$$

VIX Futures

Recall from (39) that $VIX_t^i \simeq VIX_0^i M_t^i$ where $M_t^i = \exp(Y_t^i)$ is a positive exponential martingale. The characteristic function of

$$Y_{T_i}^i = \int_0^{T_i} \eta_s^i ds + \frac{1}{2} \int_0^{T_i} \omega e^{-k_1(T_i-s)} dZ_s + \int_0^{T_i} \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx ds)$$

can be found in the same way as for X^i . It is given by

$$\begin{aligned}\mathbb{E} \left[e^{\mathrm{i}uY_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{8}\omega^2 \mathrm{i}u \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{8}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ &\quad \left. - \mathrm{i}u\lambda \int_0^{T_i} \left(\exp \left\{ \frac{1}{2}e^{-k_2(T_i-s)}m + \frac{1}{8}e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right. \\ &\quad \left. + \lambda \int_0^{T_i} \left(\exp \left\{ \frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}m - \frac{1}{8}u^2e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right\}.\end{aligned}\tag{44}$$

A.2 Exponentially Distributed Jumps

Variance Swaps

The characteristic function in (12) takes the form

$$\begin{aligned}\mathbb{E} \left[e^{\mathrm{i}uX_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{2}\omega^2 \mathrm{i}u \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{2}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ &\quad \left. - \mathrm{i}u\lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - e^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + e^{-k_2(T_i-s)}} - 1 \right) ds \right. \\ &\quad \left. + \lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - \mathrm{i}ue^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + \mathrm{i}ue^{-k_2(T_i-s)}} - 1 \right) ds \right\}.\end{aligned}\tag{45}$$

VIX Futures

Recall from (39) that $VIX_t^i \simeq VIX_0^i M_t^i$ where $M_t^i = \exp(Y_t^i)$ is a positive exponential martingale. In this specification the characteristic function of $Y_{T_i}^i$ is given by

$$\begin{aligned}\mathbb{E} \left[e^{\mathrm{i}uY_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{8}\omega^2 \mathrm{i}u \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{8}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ &\quad \left. - \mathrm{i}u\lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}e^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + \frac{1}{2}e^{-k_2(T_i-s)}} - 1 \right) ds \right. \\ &\quad \left. + \lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + \frac{1}{2}\mathrm{i}ue^{-k_2(T_i-s)}} - 1 \right) ds \right\}.\end{aligned}\tag{46}$$

B Proof of proposition 2

We can express V_t^i in (9) as

$$\begin{aligned} V_t^i &= V_0^i + \int_0^t V_{s-}^i \omega e^{-k_1(T_i-s)} dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}} V_{s-}^i (\exp \{e^{-k_2(T_i-s)} x\} - 1) J(ds \, dx) \\ &\quad - \int_0^t \int_{\mathbb{R}} V_{s-}^i (\exp \{e^{-k_2(T_i-s)} x\} - 1) \nu(dx) ds. \end{aligned}$$

Applying the Itô formula to $\sqrt{V_t^i}$ we obtain

$$\begin{aligned} \sqrt{V_t^i} &= \sqrt{V_0^i} + \frac{1}{2} \int_0^t (V_{s-}^i)^2 \omega^2 e^{-2k_1(T_i-s)} \left(-\frac{1}{4} (V_{s-}^i)^{-\frac{3}{2}} \right) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} V_{s-}^i (\exp \{e^{-k_2(T_i-s)} x\} - 1) \frac{1}{2} (V_{s-}^i)^{-\frac{1}{2}} \nu(dx) ds \\ &\quad + \int_0^t V_{s-}^i \omega e^{-k_1(T_i-s)} \frac{1}{2} (V_{s-}^i)^{-\frac{1}{2}} dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}} (V_{s-}^i)^{\frac{1}{2}} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-s)} x \right\} - 1 \right) J(ds \, dx) \\ &= \sqrt{V_0^i} + \frac{1}{2} \int_0^t \sqrt{V_{s-}^i} \omega e^{-k_1(T_i-s)} dZ_s \\ &\quad - \int_0^t \int_{\mathbb{R}} \sqrt{V_{s-}^i} \left(\frac{1}{2} (\exp \{e^{-k_2(T_i-s)} x\} - 1) - \frac{1}{8} \omega^2 e^{-2k_1(T_i-s)} \right) \nu(dx) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \sqrt{V_{s-}^i} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-s)} x \right\} - 1 \right) J(ds \, dx). \end{aligned}$$

This can be written as

$$\begin{aligned} \sqrt{V_t^i} &= \sqrt{V_0^i} \exp \left\{ \int_0^t \hat{\mu}_s^i ds + \frac{1}{2} \int_0^t \omega e^{-k_1(T_i-s)} dZ_s \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx \, ds) \right\}, \end{aligned} \tag{47}$$

where

$$\hat{\mu}_t^i = -\frac{1}{4} \omega^2 e^{-2k_1(T_i-t)} - \frac{1}{2} \int_{\mathbb{R}} (\exp \{e^{-k_2(T_i-t)} x\} - 1) \nu(dx).$$

The stochastic integrals in (47) being processes with independent increments, a straightforward application of the exponential formula for Poisson random measures e.g. Cont and Tankov (2004, Prop. 3.6.) yields the multiplicative decomposition in Proposition 2 where M is given by (37)-(38).

C Tables

Table 1: Conditional correlations between the daily returns on S&P 500 and the VIX from September 22nd, 2003 to February 27th, 2009.

Abs. Return	Unconditional	< 0.5%	> 0.5%	< 1%	> 1%	< 5%	> 5%
Correlation	-0.74	-0.45	-0.78	-0.66	-0.79	-0.76	-0.93
Observations	1368	677	691	1031	337	1348	20

Table 2: Error from following the replicating strategy relative to the realized variance over the three month windows.

	Sep-Dec 2007	Dec 2007-Mar 2008	Mar-Jun 2008	Jun-Sep 2008
Error	-0.137%	-0.025%	0.060%	-0.243%

Table 3: Calibrated parameters for the two models from the VIX volatility smiles on August 20th, 2008 together with the resulting calibration error. The top panel corresponds to the normally distributed jumps and the bottom to the double exponentially distributed jumps.

Normal jumps							
λ	ω	k_1	k_2	m	δ		Error (%)
3.5201	2.0389	21.9623	2.0743	0.5394	0.2468		0.64
Double exponential jumps							
λ	ω	k_1	k_2	p	α_+	α_-	Error (%)
13.5938	1.9765	22.3033	2.2020	0.8663	4.2457	19.9055	0.85

Table 4: Model parameters calibrated from the S&P 500 volatility smiles on August 20th, 2008 together with the resulting calibration error. The correlation between the two Brownian components set to -0.45. The second and third row in each panel correspond to the mean and variance of the jumps before scaling with $(V_{T_i}^i/V_0^i)^{\frac{1}{2}}$.

i	0	1	2	3	4	5
<u>Gaussian jumps</u>						
b_i	-0.151	-0.159	-0.152	-0.173	-0.187	-0.193
$b_i m$	-0.081	-0.086	-0.082	-0.093	-0.101	-0.104
$ b_i \delta $	0.037	0.039	0.038	0.043	0.046	0.048
Error (%)	8.8	0.6	1.1	1.8	1.9	2.7
<u>Double exponential jumps</u>						
b_i	-0.153	-0.159	-0.149	-0.169	-0.184	-0.192
$\left(\frac{b_i p}{\alpha_+} - \frac{b_i(1-p)}{\alpha_-}\right)$	-0.030	-0.031	-0.029	-0.033	-0.036	-0.038
$\left(\frac{b_i^2 p}{\alpha_+^2} + \frac{b_i^2(1-p)}{\alpha_-^2}\right)^{\frac{1}{2}}$	0.034	0.035	0.033	0.037	0.040	0.042
Error (%)	6.9	0.8	1.4	2.0	2.1	2.7

Table 5: The contribution of jumps to the forward variance swap rates. The quantity ε_i is defined in equation (42).

i	0	1	2	3	4	5
<u>Gaussian jumps</u>						
$\varepsilon_i \times 10^2$	0.100	0.119	0.133	0.157	0.225	0.252
$\frac{\varepsilon_i}{V_0^i}$ (%)	2.2	2.4	2.5	2.8	3.8	4.1
<u>Double exponential jumps</u>						
$\varepsilon_i \times 10^2$	0.105	0.120	0.131	0.157	0.218	0.252
$\frac{\varepsilon_i}{V_0^i}$ (%)	2.3	2.4	2.4	2.8	3.6	4.1

D Figures

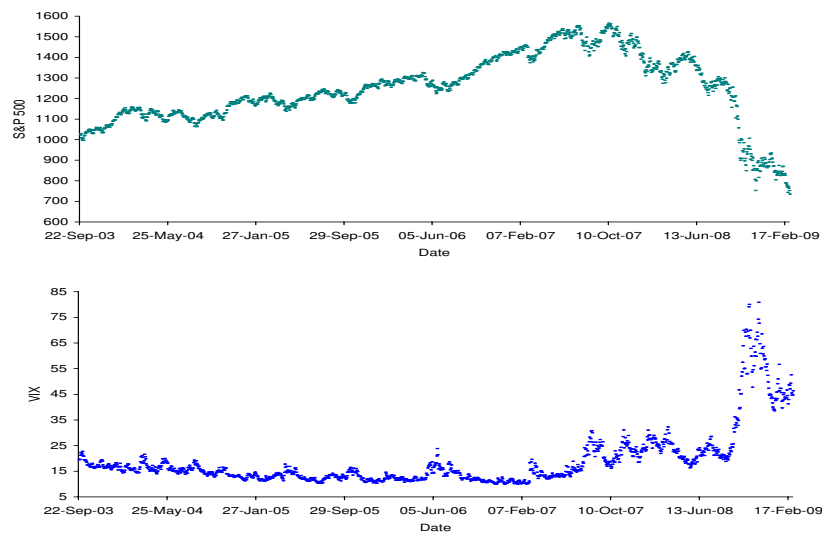


Figure 1: Time series of the VIX index (bottom) depicted together with the S&P 500 (top) covering the period from September 22nd, 2003 to February 27th, 2009.

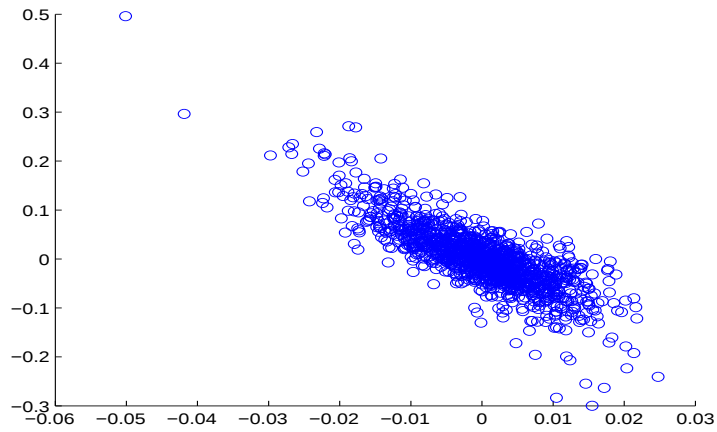


Figure 2: Daily relative changes in the VIX (vertical axis) vs daily relative changes in the S&P 500 index (horizontal axis), September 22nd, 2003 to February 27th, 2009.

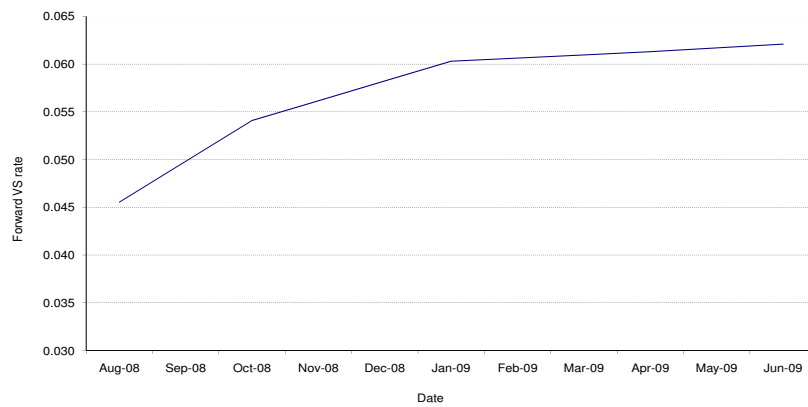


Figure 3: The term structure of 1 month forward variance swaps for the S&P 500 on August 20th, 2008.

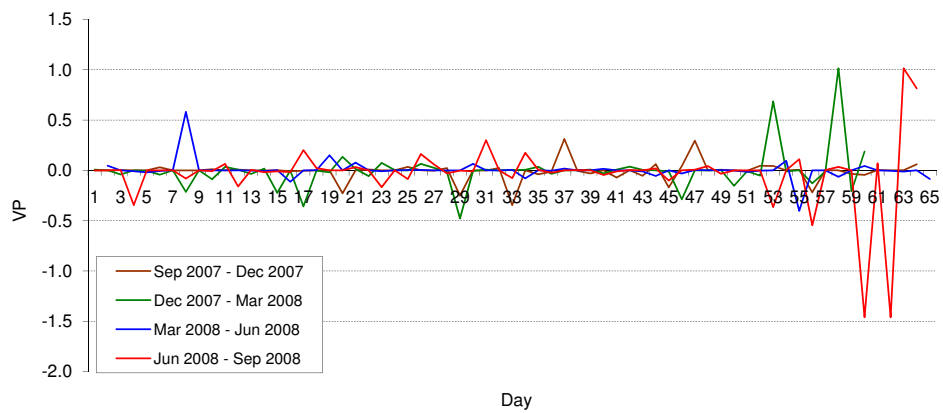


Figure 4: The profit-loss from day to day in variance points (VP) from holding a long position of the hedge portfolio and a short variance swap for four different three month variance swaps. The x -axis is trading days during the three month period.

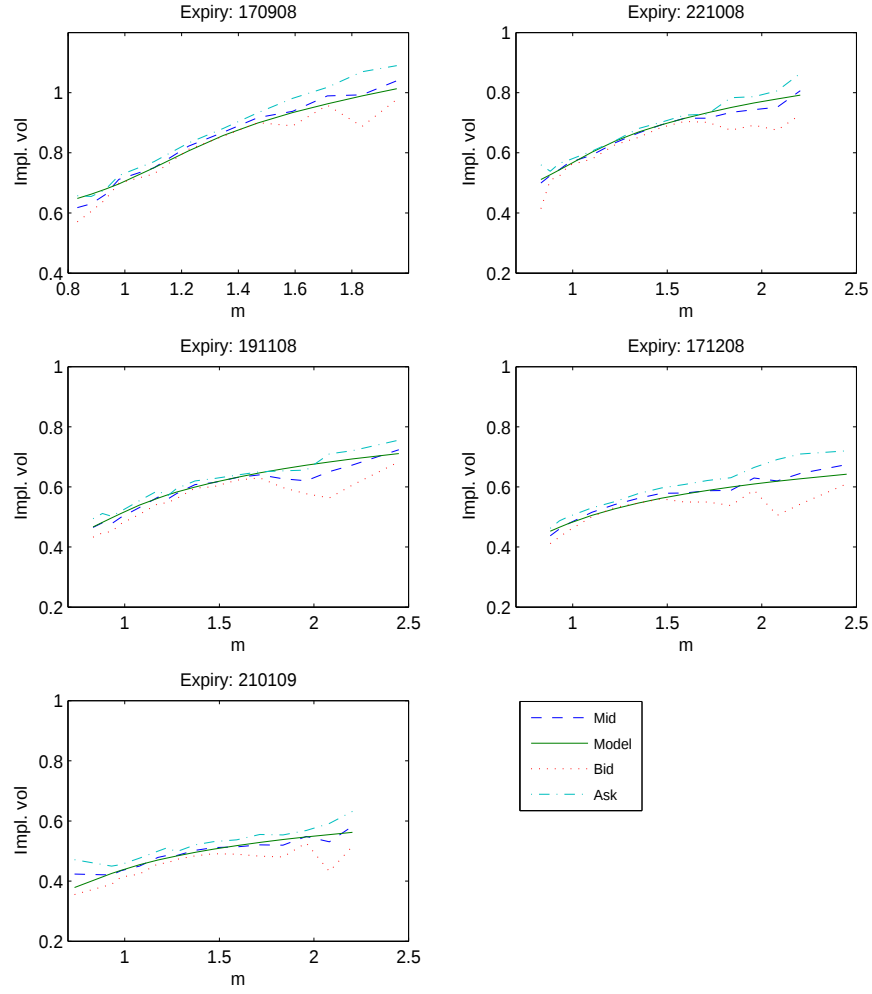


Figure 5: VIX implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the x axis.

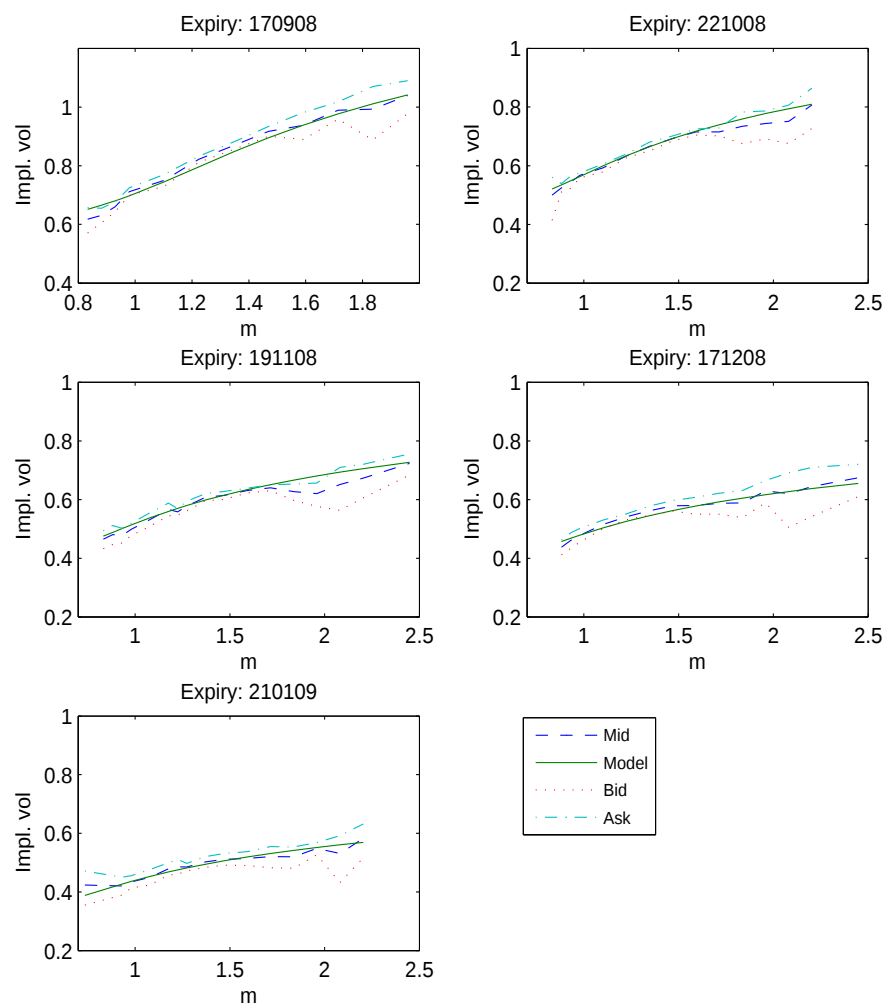


Figure 6: VIX implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the x axis.

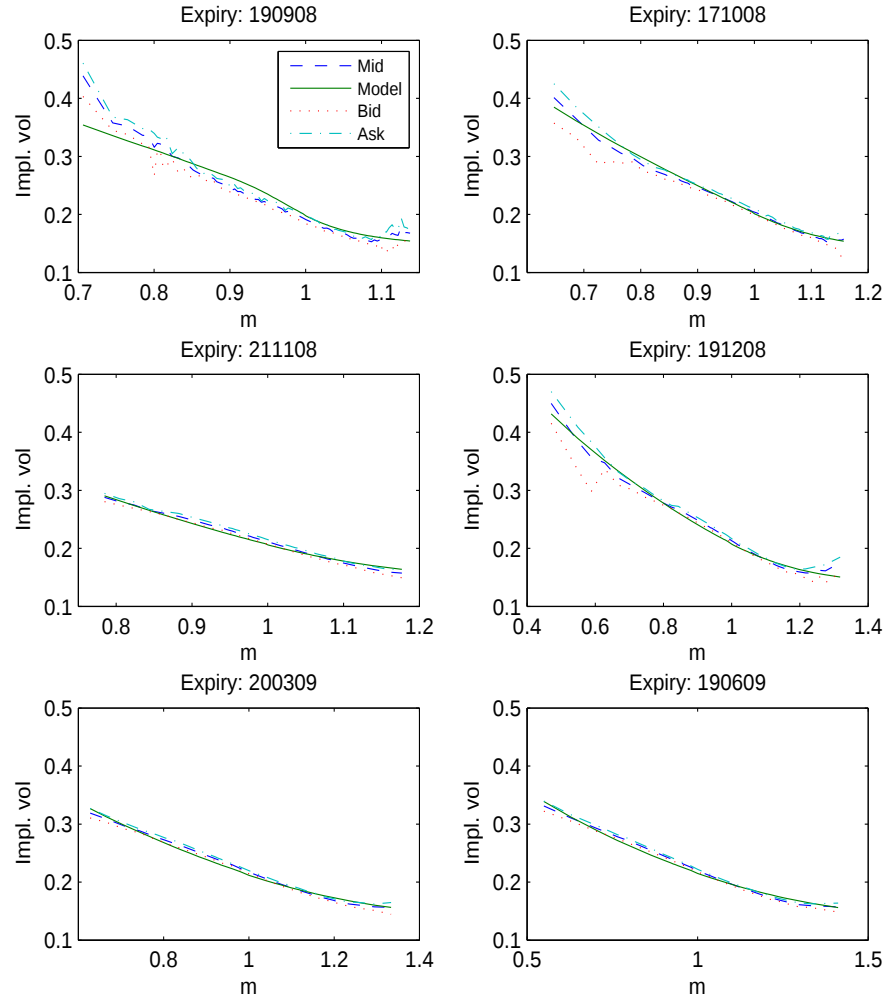


Figure 7: S&P 500 implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/S_t$ on the x axis.

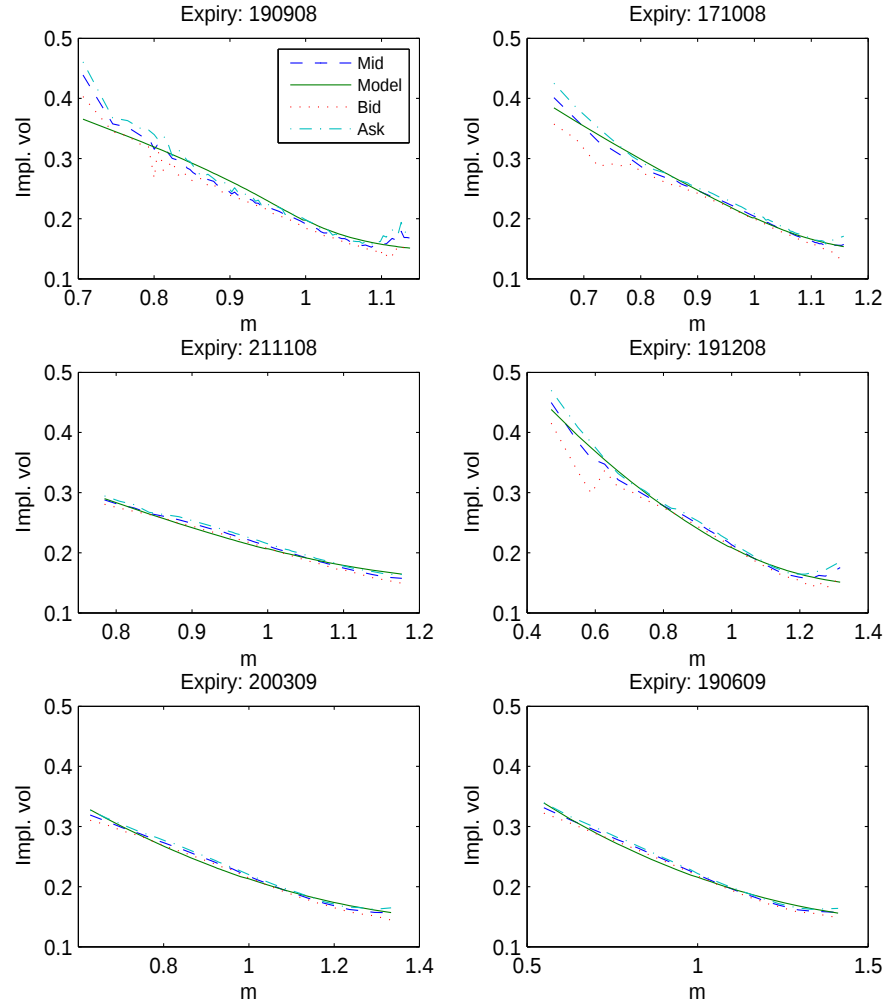


Figure 8: S&P 500 implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against moneyness $m = K/S_t$ on the x axis.