Dürer's Unfolding Problem for Convex Polyhedra

Mohammad Ghomi

Convex polyhedra are among the oldest mathematical objects. Indeed the five platonic solids, which constitute the climax of Euclid's books, were already known to the ancient people of Scotland some 4,000 years ago; see Figure 1. During the Renaissance, polyhedra were once again objects of fascination while painters were discovering the rules of perspective and laying the foundations of projective geometry. This remarkable confluence of art and mathematics was personified in a number of highly creative individuals including the German painter Albrecht Dürer, who was based in Nüremberg at the dawn of the sixteenth century and is credited with ushering in the Renaissance in Northern Europe. During extended trips over the Alps, Dürer learned the rules of perspective from his Italian contemporaries, and he subsequently described them in his influential book, *The Painter's Manual*. Aside from being the first geometry text published in German, this work is remarkable for containing the first recorded examples of unfoldings of polyhedra; see Figure 2.

remarkable confluence of art and mathematics

An (edge) unfolding of a polyhedron P is the process of cutting it along a collection of its edges, without disconnecting it, so that the resulting surface may be developed isometrically into the plane. Many school children are familiar with the process of cutting out a template from craft books, and folding the paper along dotted lines to form simple polyhedra such as a tetrahedron or a cube; an unfolding is the reverse process. Note that the cuts are made along a connected subset

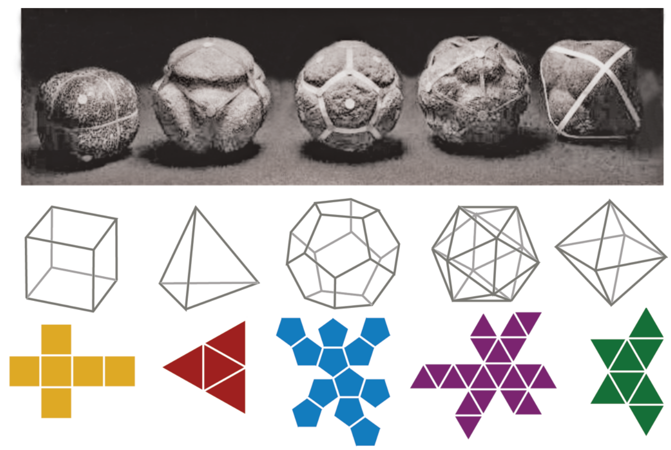

Figure 1. First Row: Neolithic carved stones from 2000 BC discovered in Scotland.[1] [1]
Second Row: The familiar representations of platonic solids studied in Euclid's *Elements*.
Third Row: Examples of unfoldings of the platonic solids.

Figure 2. A self-portrait of Dürer completed in the year 1500 at the age of twenty-eight, together with some illustrations from his book, *The Painter's Manual*, including the first recorded examples of unfoldings.

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[1] *For more information see* Time Stands Still: New Light on Megalithic Science *by Keith Critchlow.*

of the edges of P which contains each vertex of P and no closed paths. In other words, the cut set forms a *spanning tree* of the edge graph of P, and thus a convex polyhedron admits many different unfoldings depending on the choice of this tree. Furthermore, it is not the case that every unfolding of every polyhedron is simple or nonoverlapping. For instance there are even some (nonregular) tetrahedra which admit some unfoldings that overlap themselves. On the other hand, all the examples of unfoldings which Dürer constructed were simple, and in the intervening five centuries no one has yet discovered a convex polyhedron which does not admit some simple unfolding.

The problem of existence of simple unfoldings for convex polyhedra was explicitly posed in the 1970s by Shephard, and the assertion that a solution can always be found, or that every convex polyhedron is unfoldable (in one-to-one fashion) has been dubbed Dürer's conjecture. There is, however, substantial empirical evidence both for and against this supposition. On the one hand, computers have found simple unfoldings for countless convex polyhedra through an exhaustive search of their spanning edge trees. On the other hand, there is still no algorithm for finding the right tree, and computer experiments [4] suggest that the probability that a random edge unfolding of a generic polyhedron overlaps itself approaches 1 as the number of vertices grows. To date the problem remains wide open, and it is not even known whether simple classes of polyhedra such as prismatoids (polyhedra generated by the convex hull of a pair of convex polygons in parallel planes) are unfoldable.

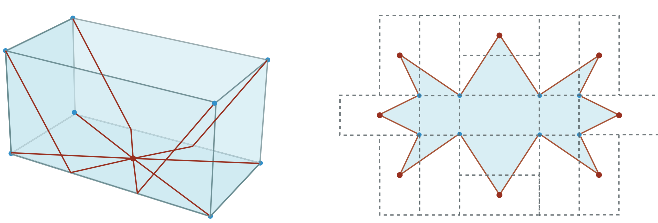

Figure 3. A. D. Alexandrov observed that one may develop a convex polyhedron injectively into the plane by cutting along geodesics which connect the vertices to a generic point.

If one is not confined to cut only along the edges, then it is quite easy to develop a polyhedron into the plane in one-to-one fashion, as had been observed by the influential geometer A. D. Alexandrov in his seminal work *Convex Polyhedra*; see Figure 3. Why then is Dürer's problem so difficult? Perhaps because the edges of convex polyhedra are not well understood, in the sense that there is no known procedure or simple algorithm for detecting an edge by means of intrinsic measurements within the surface. Indeed, Alexandrov's embedding theorem for convex surfaces—which states that any locally convex polyhedral metric on the sphere $\mathbf{S}^2$ may be realized as a convex polyhedron in Euclidean space $\mathbf{R}^3$—is not constructive and gives no hint as to which geodesics between a pair of vertices are realized as edges. In 2008, a more constructive proof was given by Bobenko and Izmestiev; however, this proof does not specify the location of the edges either.

The edge graph of a convex polyhedron is not the unique 3-connected embedded graph in the polyhedron whose vertices coincide with those of the polyhedron, whose edges are distance minimizing geodesics, and whose faces are convex. It seems reasonable to expect that Dürer's conjecture should be true if and only if it holds for this wider class of *pseudo-edge graphs*. This approach was studied by Alexey Tarasov in 2008, and has been further investigated by the author and Nicholas Barvinok very recently [2]. We claim to have constructed a convex polyhedron with 176 vertices and a pseudo-edge graph which does not admit any nonoverlapping unfolding. Thus one may say that Dürer's conjecture does not hold in a purely intrinsic sense.

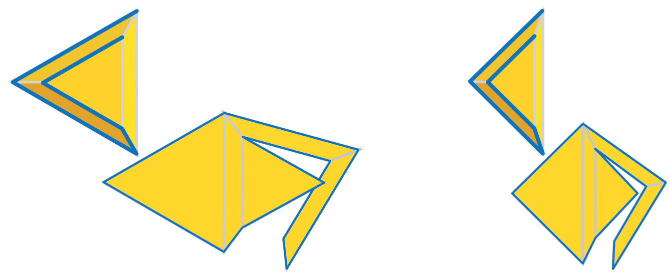

Figure 4. The left side shows a truncated tetrahedron (viewed from above) together with an overlapping unfolding of it generated by a monotone edge tree. As we see on the right side, however, the same edge tree generates a simple unfolding once the polyhedron has been stretched.

On the other hand, in 2014 the author [5] had obtained a positive result in this area by solving a weaker form of Dürer's problem posed by Croft, Falconer, and Guy [3, B21]: is every convex polyhedron combinatorially equivalent to an unfoldable one? It turns out that the answer is yes, and therefore there exists no combinatorial obstruction to a positive resolution of Dürer's problem. What the author shows is that every convex polyhedron becomes unfoldable after an affine (or linear) transformation. More explicitly, suppose that a convex polyhedron P is in general position in $\mathbf{R}^3$, i.e., no two of its vertices are at the same height. Then it is easy to construct a spanning tree T of P which is *monotone*, i.e., if T is rooted at the lowest vertex r of P, then the branches of T which connect its leaves to r have strictly decreasing heights or z-coordinates. Now stretch P via a rescaling along the z-axis. Then the corresponding unfolding eventually becomes simple, as illustrated in Figure 4. The proof that this stretching procedure works is by induction on the number of leaves (or branches of T which connect each leaf to the root r). The first step, i.e., when T consists of only one branch, is relatively simple to prove and follows from a topological characterization for embeddings among immersed disks in the plane. The inductive step is more technical.

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ABOUT THE AUTHOR

Mohammad Ghomi works primarily on classical problems involving curves and surfaces in Euclidean space, ranging from differential geometry and topology to real algebraic geometry and combinatorics.

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