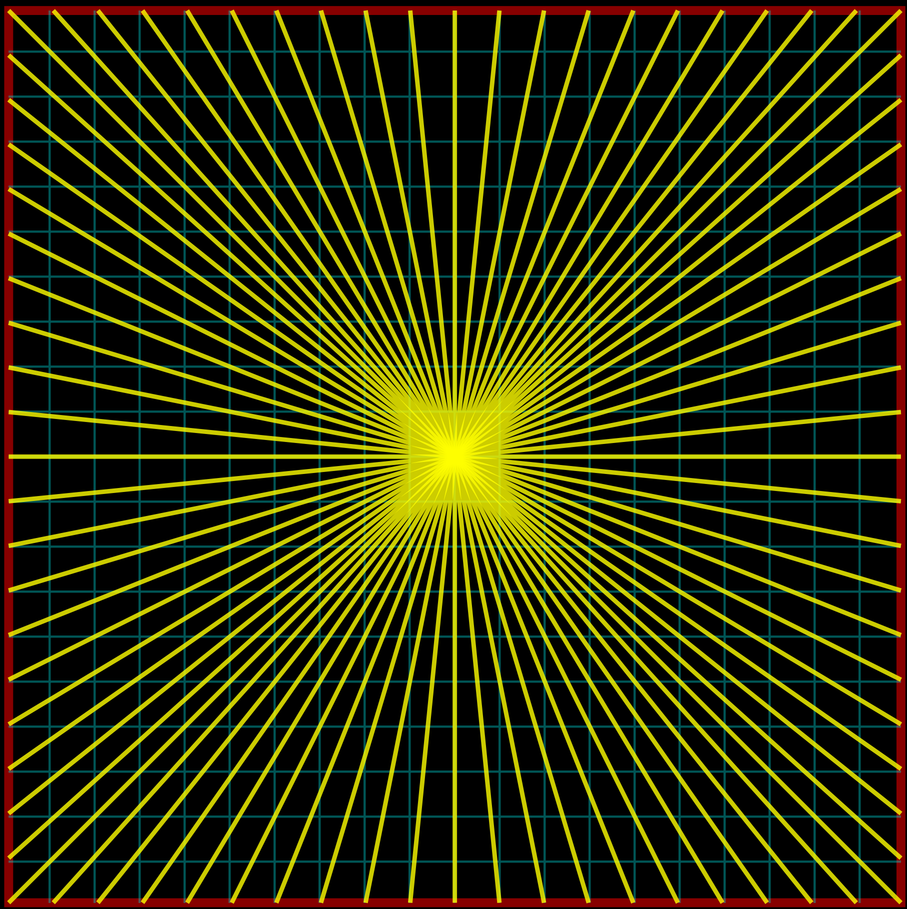
Concurrent lines

We create rays in a square in the following way. Make a big square with a square grid inside. Now draw lines from the center of the big square to the intersection points of the square grid and your big square.



For such rays in a square we have stated that the resulting moiré pattern by superimposing a horizontally shifted second version are parabolas and horizontal lines and illustrated it in a GeoGebra animation: (.html). For the description of the lines there are two cases, the first for lines below $y = x$ and the second above.

First case

The moiré pattern of two groups of concurrent lines is derived using the indicial equation method. The lines of the first group all intersect at the origin, while the second group is horizontally shifted by a distance s.

Let's consider the two groups of concurrent lines:

- The first group of lines is described by the equation $y = mx$.
- The second group of lines is described by the equation $y = n(x - s)$.

Here, m and n are integer indices. They change the height in the slope triangle by one unit each. As you can see in my GeoGebra simulation, m and n would typically be divided by a fixed number that I call p there. However, for the derivation, it has no significance, and I leave it away for simplicity here.

The indicial equation, relating the parametrisation of the two line bundles is

$$m - n = k$$

where k is an integer that indexes the resulting moiré pattern.

Derivation

Step 1: Solve for the indices m and n from the line equations:

$$m = \frac{y}{x}$$

$$n = \frac{y}{x - s}$$

Step 2: Substitute these expressions into the indicial equation:

$$\frac{y}{x} - \frac{y}{x - s} = k$$

Step 3: Factor out y and solve for the final equation:

$$y\left(\frac{1}{x} - \frac{1}{x - s}\right) = k$$

To simplify the expression in the parentheses, find a common denominator, which is $x(x - s)$.

$$\frac{1}{x} - \frac{1}{x - s} = \frac{x - s}{x(x - s)} - \frac{x}{x(x - s)} = \frac{x - s - x}{x(x - s)} = \frac{-s}{x(x - s)}$$

Substitute this simplified term back into the equation:

$$y\left(\frac{-s}{x(x - s)}\right) = k$$

Now, isolate y to get the final equation for the moiré pattern:

$$y = k\left(\frac{x(x - s)}{-s}\right)$$

$$y = -\frac{k}{s}(x^2 - sx)$$

$$y = -\frac{k}{s}x^2 + kx$$

Result

The final equation $y = -\frac{k}{s}x^2 + kx$ is a **quadratic function**, which is the standard form of a parabola. This confirms that the moiré pattern formed by superimposing our two groups of concurrent lines leads to a family of **parabolas**.

Second case

Indicial equation and derivation

We begin with the two sets of concurrent lines:

- $y = \frac{x}{m} \Rightarrow m = \frac{x}{y}$
- $y = \frac{x-s}{n} \Rightarrow n = \frac{x-s}{y}$

The integers m and n now change the width in the slope triangle by one unit each. As you can see in my GeoGebra simulation (.html), m and n would typically divide a fixed number that I call p there. For simplicity, we leave it out again.

Now, we substitute these expressions into the indicial equation $m - n = k$:

$$\frac{x}{y} - \frac{x-s}{y} = k$$

Since the terms have a common denominator (y), we can simplify the expression:

$$\frac{x-(x-s)}{y} = k$$
$$\frac{s}{y} = k$$

Finally, we isolate y to get the equation for the moiré pattern:

$$y = \frac{s}{k}$$

Result

The final equation, $y = \frac{s}{k}$, is the equation of a **horizontal line**. This confirms that with these starting conditions, the moiré pattern formed is a family of **parallel lines**.

Family of Lines under Square Inversion: (.pdf)

Some other derivations with the “Indicial Equation Method”: (.pdf)
